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Periodic cyclic homology of smooth crossed product algebras

A Dirac-dual Dirac method in periodic cyclic homology

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Résumé et mots clés

Cette thèse est dédiée à l'étude des algèbres de convolution lisses associées aux groupes de Lie, apparaissant naturellement en géométrie non-commutative et en théorie des représentations. Notre résultat établit un isomorphisme, après stabilisation, entre l'homologie cyclique périodique des algèbres produits croisés lisses associées à un groupe de Lie réel et à son sous groupe compact maximal, produisant un analogue homologique de l'induction de Dirac à coefficients pour les groupes de Lie.

Notre approche repose sur les travaux de Nistor qui établit cet isomorphisme localement, c'est-à-dire au-dessus de chaque classe de conjugaison du groupe, et sans stabilisation. Les outils principaux utilisés proviennent de la K-théorie bivariante de Kasparov et de sa description par Cuntz. Nous proposons un raffinement de la méthode de Dirac-dual Dirac et de la notion d'équivalence de Morita pour une adaptation à notre cadre.

Mots clés : Géométrie non-commutative, Algèbres de convolution, Homologie cyclique, K-théorie, K-théorie bivariante, Groupes de Lie, Théorie des représentations

Abstract and keywords

This thesis is devoted to the study of smooth convolution algebras associated to Lie groups, appearing naturally in non-commutative geometry and representation theory. Our result establishes a stable isomorphism between the periodic cyclic homology of the smooth crossed product algebras associated to a real Lie group and to its maximal compact subgroup. It may be viewed an homological analogue to the Dirac induction with coefficients for Lie groups.

Our approach is built on earlier work of Nistor who established this isomorphism locally, *i.e.* around each conjugacy class of the group and without stabilization. Essential tools come from Kasparov's bivariant K-theory and its interpretation by Cuntz. We set up a refinement of the Dirac-dual Dirac method and of Morita equivalence which fit into our framework.

Keywords: Non-Commutative geometry, Convolution algebras, Cyclic homology, K-theory, bivariant K-theory, Lie groups, Representation theory

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Introduction

The starting point of our study is the interplay between representations of groups and representations of algebras. For a locally compact group G, the convolution algebra $\mathscr{C}_c(G)$ of compactly supported continuous functions provides a natural algebraic model. Representations of the group correspond to modules over this algebra, allowing one to pass from group-theoretic data to algebraic structures. However, this algebraic viewpoint alone is not sufficient to capture the analytic features of representations. This leads naturally to the construction of group $C^\star$-algebras, obtained as completions of $\mathscr{C}_c(G)$ with respect to suitable norms. The full group $C^\star(G)$ encodes all unitary representations, while the reduced group $C^\star_r(G)$ captures those that are tempered. These constructions lie at the heart of non-commutative geometry, they encode the harmonic analysis of a group through a non-commutative algebra.

Furthermore, the additional data of a group action on a locally compact space (and more generally on a locally convex algebra) yields to the notion of convolution algebra *with coefficients*, i.e. crossed product algebras. These can be viewed as a noncommutative analogue of the quotient of a space by a group action and provide a natural framework for studying orbit spaces. In our case of interest, the group G will be supposed of Lie-type and its action on a locally convex algebra A will be supposed smooth. It gives rise to the smooth convolution algebra $A \rtimes G = \mathscr{C}^\infty_c(G, A)$ with A-valued coefficients. The aim of this thesis is the study of homological invariants (in particular periodic cyclic homology) of this convolution algebra.

Denote by K a maximal compact subgroup of a real Lie group G. The associated homogeneous space $G/K \approx \mathbb{R}^m$ is contractible and one might expect a link between the homological invariants of $A \rtimes G$ and of $A \rtimes K$. The famous Baum-Connes conjecture (with coefficients) claims in the case of Lie groups the existence of an isomorphism up to a degree shift

$$K_\bullet(A \rtimes_r K) \xrightarrow{\sim} K_{\bullet+m}(A \rtimes_r G) \tag{1}$$

between the K-groups of the reduced $C^\star$-crossed products by G and K with respect to a $C^\star$-algebra A endowed with a continuous action of G. In the early 90-s, V. Nistor established a result similar to conjecture (1) for the periodic cyclic homology of smooth crossed products. He constructed an isomorphism

$$HP_\bullet(A \rtimes K)_{\langle x\rangle} \simeq HP_{\bullet+m}(A \rtimes G)_{\langle x\rangle}. \tag{2}$$

where the subscript denotes the localization at the maximal ideal of the algebra of class functions on G given by functions vanishing at the conjugacy class $\langle x\rangle$. However, these local isomorphisms cannot be glued together to obtain a global correspondence between the homology groups of $A \rtimes K$ and $A \rtimes G$.

In this thesis, we construct a global isomorphism (after stabilization) between these periodic cyclic homology groups, thereby providing a homological analogue of the Baum–Connes conjecture for Lie groups with coefficients. Our approach decomposes this isomorphism into two different pieces via an intermediate space :

$$HP_\bullet(A \rtimes K) \xrightarrow{\sim} HP_\bullet(\mathscr{C}_c^\infty(G/K, A) \rtimes G) \xrightarrow{\sim} HP_{\bullet+m}(A \rtimes G). \tag{3}$$

The left-hand isomorphism is well-known at the $C^\star$-algebraic level and requires some adaptations to fit the Fréchet algebras framework, while the right-hand isomorphism, way more complicated, follows the strategy below.

In the easiest case where $A = \mathbb{C}$ is the trivial algebra, the central part of the right-hand isomorphism relies on the identification of $HP_\bullet(\mathscr{C}_c^\infty(G/K)) = HP_\bullet(\mathscr{C}_c^\infty(\mathbb{R}^m))$ with $HP_{\bullet+m}(\mathbb{C})$. Our first idea was to obtain this identification adapting the Thom isomorphism (Poincaré duality) occurring at the level of De Rham cohomology with compact support. This first approach did not work, however, because the constructions we could obtain were not compatible with crossed products. A key observation is that periodic cyclic homology also admits a close relationship with K-theory, where a Thom isomorphism (Bott periodicity) holds :

$$K_\bullet\left(\mathscr{C}_c^\infty(\mathbb{R}^m)\right) \simeq K^\bullet\left(\mathbb{R}^m\right) \xrightarrow{\sim} K^{\bullet+m}\left(\{pt\}\right) \simeq K_{\bullet+m}(\mathbb{C}).$$

It can be proved using the *Dirac-dual Dirac method* at the level of equivariant Kasparov's bivariant K-theory, which is now compatible with crossed products. Our strategy for the right-hand isomorphism of (3) then relies on the adaptation of the Dirac-dual Dirac method from the Kasparov bivariant K-theory to our needs.

Introduced by G. Kasparov, the bivariant K-theory assigns to a pair of $C^\star$-algebras (A, B) an abelian group $KK(A, B)$. The group $KK(A, \mathbb{C})$ is the group of homotopy classes of Fredholm modules over A and stands as a non-commutative analogue of elliptic operators. The group $KK(\mathbb{C}, B)$, on the other hand encodes idempotents and invertible matrices over B, which play the role of vector bundles over a locally compact space. The main feature of the theory is the associative Kasparov product

$$- \times - : KK(A, B) \times KK(B, C) \longrightarrow KK(A, C). \tag{4}$$

If $A = C = \mathbb{C}$ and $B = \mathscr{C}(M)$ is the algebra of continuous functions over a compact space M, the Kasparov product recovers the index pairing between elliptic operator and vector bundles over M. Kasparov also developed an equivariant version of this theory which preserves the product (4).

Consider a Lie group G with maximal compact subgroup K as before. Kasparov constructed two distinguished classes

$$\alpha \in KK_G(\mathscr{C}_\tau(G/K),\mathbb{C}) \ \text{ and } \ \beta \in KK_G(\mathbb{C},\mathscr{C}_\tau(G/K)),$$

called *Dirac element* and *dual Dirac element*, where $\mathscr{C}_\tau(G/K)$ is a certain algebra of *"matrix valued functions"* on G/K. They satisfy the fundamental identities :

$$\alpha \times \beta = 1 \in KK_G(\mathscr{C}_\tau(G/K),\mathscr{C}_\tau(G/K)) \ \text{ and } \ \mathrm{Res}_K^G(\beta\times\alpha) = 1 \in KK_K(\mathbb{C},\mathbb{C}) = \mathrm{Rep}(K).$$

These relations make α and β as dual to each others in a K-equivariant way. Our aim will be to construct some analogues of these Dirac and dual Dirac elements at the level periodic cyclic homology which satisfy similar relations.

For that, we need a more *algebraic* description of the bivariant K-theory. It is provided by the Cuntz' picture which identifies bivariant K-theory with homotopy classes of $C^\star$-algebra homomorphisms between the Cuntz algebra qA and the algebra B stabilized by compact operators :

$$KK(A,B) \xrightarrow{\sim} \mathrm{Hom}(qA, B\otimes_\pi \mathscr{K})_{/\text{homotopy}}. \tag{5}$$

In this setting, the Kasparov product corresponds essentially to the composition of associated $C^\star$-algebras homomorphisms. In this thesis we present an adaptation of the right-hand side of this identification to fit the Banach/Fréchet context. The idea to use the Cuntz-picture of bivariant K-theory to obtain a bivariant Chern character is due to Nistor [Nis93] [Nis91]. We use here a slightly different approach, making heavily use of excision [CQ95], which was not yet known in the early nineties.

Our first idea is to propose a rescaling model for the Cuntz algebra qA, which is defined as an ideal of the $C^\star$-free product QA. For any $R>0$, we introduce a parametrized version of a free product, denoted Q_RA, as a completion of the algebraic free product by a rescaled norm. The associated ideal q_RA becomes the main receptacle of differences of Fréchet algebra homomorphisms out of A, of norm smaller than R. As in [Cun87], we show that there exists a canonical and natural homotopy equivalence between Q_RA and the direct sum $A\oplus A$ after stabilization by square matrices. It descends as a homotopy equivalence

$$q_RA \xrightarrow{\sim} A$$

which is natural, stable under group actions and compatible with crossed product constructions.

The second idea suggests a refinement of the algebra of compact operators $\mathscr{K}$ appearing in the Cuntz-picture (5) using Schatten ideals, more compatible with periodic cyclic homology. We show that for $p>\dim(G/K)$, there exists a parameter $R>0$ great enough so that the Dirac and dual Dirac elements induce Fréchet algebra homomorphisms :

$$\alpha^\sharp : q_R(\mathscr{C}_\tau^\infty(G/K)) \longrightarrow \ell^{p,G} \quad \text{and} \quad \beta^\sharp : q_1\mathbb{C} \longrightarrow M_2(\mathscr{C}_\tau^\infty(G/K)), \tag{6}$$

where $\mathscr{C}_\tau^\infty(G/K)$ is a smooth version of the matrix valued algebra $\mathscr{C}_\tau(G/K)$ and $\ell^{p,G}$ is the algebra of G-smooth vectors in the p-th Schatten ideal.

The third idea consists on defining a notion of equivalence for G-algebras which makes these algebraic constructions compatible with crossed products at the level of periodic cyclic homology. Refining an idea of Cuntz and Quillen [CQ95], we introduce the notion of quasi-Morita equivalence with the property that if A and B are quasi-Morita equivalent G-algebras, then there exists a chain-homotopy equivalence between the periodic cyclic complexes of their associated crossed products. As a Fréchet analogue of the Green's Imprimitivity theorem, we establish the quasi-Morita equivalences $A \sim \mathscr{C}_c^\infty(G,A) \rtimes G$ and $\mathscr{C}_c^\infty(G/K,A) \sim \mathscr{C}_c^\infty(G,A) \rtimes K$ which establishes the left-hand piece of (3) :

$$HP_\bullet(A \rtimes K) \xrightarrow{\sim} HP_\bullet(\mathscr{C}_c^\infty(G/K,A) \rtimes G).$$

Moreover, the quasi-Morita equivalences $\mathscr{C}_c^\infty(G/K) \sim \mathscr{C}_\tau^\infty(G/K)$ and $\ell^1 \sim \mathbb{C}$ imply that the Dirac and dual Dirac homomorphisms of (6) descend as chain-complex homomorphisms

$$(\alpha^\sharp \otimes id_{\mathscr{A}}) \rtimes G : \widehat{CC}(\mathscr{C}_c^\infty(G/K,\mathscr{A}) \rtimes G) \longrightarrow \widehat{CC}(\mathscr{A} \rtimes G)[\dim(G/K)],$$

$$(\beta^\sharp \otimes id_{\mathscr{A}}) \rtimes G : \widehat{CC}(\mathscr{A} \rtimes G) \longrightarrow \widehat{CC}(\mathscr{C}_c^\infty(G/K,\mathscr{A}) \rtimes G)[\dim(G/K)],$$

where $\mathscr{A} = A \otimes_\pi \mathscr{D}$ denotes the stabilization of A by a certain G-smooth Schatten ideal. The right-hand piece of (3) relies on the fact that these morphisms are homotopy inverse to each others. The proof of this statement follows two steps. First of all, the fundamental identities (6) show that α and β are dual to each others in a K-equivariant way and the Cuntz-picture (5) relates this Kasparov product to the composition of the associated K-crossed products homomorphisms, which implies that

$$(\alpha^\sharp \otimes id_{\mathscr{A}}) \rtimes K : HP_\bullet(\mathscr{C}_c^\infty(G/K,\mathscr{A}) \rtimes K) \xrightarrow{\sim} HP_{\bullet+\dim(G/K)}(\mathscr{A} \rtimes K)$$

is an isomorphism of inverse $(\beta^\sharp \otimes id_{\mathscr{A}}) \rtimes K$. Secondly, the main theorem of Nistor (2), which identifies the periodic cyclic homology of crossed products by G and K around each conjugacy class of G, implies that $(\alpha^\sharp \otimes id_{\mathscr{A}}) \rtimes G$ is a quasi-isomorphism *on each stalk*. Since it is globally-defined, it induces a global isomorphism

$$(\alpha^\sharp \otimes id_{\mathscr{A}}) \rtimes G : HP_\bullet(\mathscr{C}_c^\infty(G/K,\mathscr{A}) \rtimes G) \xrightarrow{\sim} HP_{\bullet+\dim(G/K)}(\mathscr{A} \rtimes G)$$

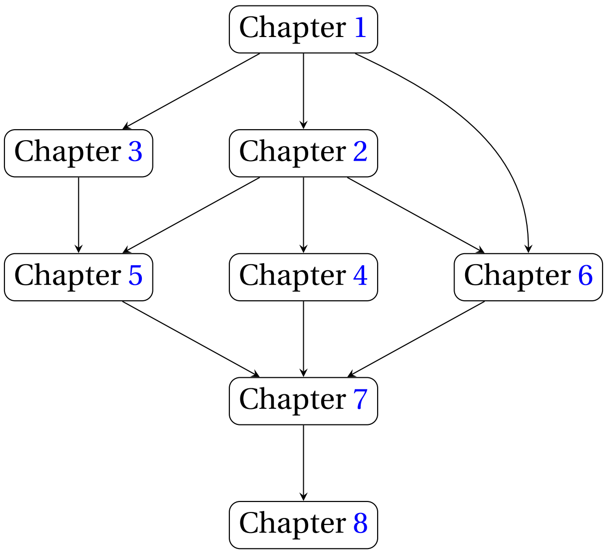
of inverse $(\beta^\sharp \otimes id_{\mathscr{A}}) \rtimes G$, establishing the right-hand side of (3). The combination of all these arguments shows the main theorem of this thesis.

Theorem. *Let G be a real Lie group with maximal compact subgroup K. For every* *complete locally convex algebra A endowed with a smooth action of G, the Dirac element* $\alpha \in KK_G(\mathscr{C}_\tau(G/K), \mathbb{C})$ *realizes the following isomorphism :*

$$HP_\bullet(\mathscr{A} \rtimes K) \underset{\sim}{\xrightarrow{\text{Morita}}} HP_\bullet(\mathscr{C}_c^\infty(G/K, \mathscr{A}) \rtimes G) \underset{\sim}{\xrightarrow{(\alpha^\sharp \otimes id_{\mathscr{A}}) \rtimes G}} HP_{\bullet + \dim(G/K)}(\mathscr{A} \rtimes G).$$

where $\mathscr{A} = A \otimes_\pi \mathscr{D}$ *is the stabilization of A with a certain G-smooth Schatten ideal.*

The thesis is organized as follows. In Chapter 1, we introduce convolution algebras, group representations, and crossed product constructions. Chapter 2 reviews K-theory, both in its algebraic and topological forms, and introduces the tools needed for later developments, especially the dual Dirac element. Chapter 3 is devoted to index theory and K-homology, providing the conceptual background for the definition of the Dirac element. In Chapter 4, we develop periodic cyclic homology and its relation to K-theory via the Chern character. Chapter 5 introduces bivariant K-theory and its Cuntz approach. Chapter 6 discusses the Baum-Connes conjecture and related results, i.e. the key motivation for our study. Chapter 7 proposes a notion of Morita equivalence that produces chain-homotopy equivalences at the level of periodic cyclic homology. Finally, Chapter 8 presents the main results of the thesis, including the construction of Dirac and dual Dirac elements in a Banach framework and their associated Dirac-dual Dirac method at the level of periodic cyclic homology.



Chapters dependency diagram

1 Representations and crossed product algebras

Sommaire

This chapter develops the analytic and algebraic framework underlying crossed product constructions. We begin with convolution algebras associated to locally compact groups, emphasizing their role as algebraic models for representation theory. The passage from convolution algebras to their $C^\star$-completions allows one to encode different classes of representations, notably unitary and tempered representations. We then introduce crossed product algebras, which generalize group $C^\star$-algebras by incorporating an action on an auxiliary algebra. These constructions provide a noncommutative model for dynamical systems and play a central role in noncommutative geometry. Finally, we discuss smooth versions of these algebras, which retain finer analytic information and are better suited for Banach homological methods. These smooth crossed products will serve as the main objects of study in later chapters.

1.1 Convolution algebras

The convolution algebra $\mathscr{C}_c(G)$ of a locally compact group G provides an algebraic framework for representation theory : instead of considering individual group elements, one studies linear combinations acting via convolution. This leads naturally to group $C^\star$-algebras, obtained by completing $\mathscr{C}_c(G)$ in suitable norms. The full group $C^\star(G)$ encodes all unitary representations, while the reduced group $C_r^\star(G)$ captures the tempered representations. These constructions link harmonic analysis and operator algebras. Classical examples include for instance finite groups (recovering group algebras) and abelian groups (via Fourier transform). This section introduces convolution and completions for the purpose of representation theory.

1.1.1 Group representations

We set G to be a topological group. A (complex) **representation** of G is the data of a complex Hilbert space V with a group homomorphism $\pi : G \to \mathrm{End}(V)$ such that $(g,v) \mapsto \pi(g)(v)$ is continuous for the topology of G and V. We will write π, V or (π, V) depending on the context. A **morphism of representations** between (π, V) and (τ, W) is a complex-linear map $\phi : V \to W$ such that for all $g \in G$

$$\phi \circ \pi(g) = \tau(g) \circ \phi.$$

The space of representations of G defines a category $\mathrm{Rep}(G)$ which encapsulates deep information coming from the group and is a key structure to study it. In good circumstance, it is even possible to recover the group from its representation category as state *Pontryagin* or *Tannaka-Krein* dualities [Bru94]. Given two representations (π, V) and (τ, W), their **direct sum** $V \oplus W$ and their **tensor product** $V \otimes W$ are representations of G for the rules :

$$(\pi \oplus \tau)(g) := \pi(g) \oplus \tau(g) \text{ and } (\pi \otimes \tau)(g) := \pi(g) \otimes \tau(g).$$

The space of *finite dimensional representations* of G is then a ring called **representation ring of** G and denoted $R(G)$. When $G = \Gamma$ is discrete, $R(\Gamma) = \mathbb{C}[\Gamma]$ is the classical algebra of complex valued Dirac functions on Γ. Also, we say that a representation (π, V) is **irreducible** if any closed subspace $W < V$ which is stable under $\pi(g)$ for all $g \in G$ is either $W = \emptyset$ or $W = V$.

Theorem 1.1.1. *When $G = \Gamma$ is discrete, the algebra $\mathbb{C}[\Gamma]$ realizes the following equivalence of categories :*

$$\mathrm{Rep}(\Gamma) \simeq \mathbb{C}[\Gamma] - \mathrm{mod}.$$

This result motivates the study of convolution algebras as they encapsulate, at least for the discrete case, the major behavior of the category of representations. It stands as an non-commutative analogy of classical geometry : whereas the study of commutative algebras provides information on the *spaces of points*, non-commutative algebras describe the *spaces of representations*.

1.1.2 Convolution algebra

Classical geometry associates to a space the commutative algebra of continuous functions vanishing at infinity. This algebra encapsulates all the geometrical information arising from the space. Now, for a group (discrete, compact, locally compact, p-adic, etc) we want to recover the same process : to associate a canonical algebra outlining the behavior of the group. The answer is multiple, depending on the topology of the group and the behavior we want to extract from it, but they all appear as completions of the *convolution algebra* of the group.

In this setup, the basic non-commutative algebra associated to G is the **convolution algebra** $\mathscr{C}_c(G)$ of compactly supported complex valued continuous functions on G [Bou06]. Fixing a Haar measure on G, the convolution algebra is endowed with the product :

$$(f_1 \star f_2)(g) = \int_G f_1(s) f_2(s^{-1}g)\,ds.$$

When $G = \Gamma$ is discrete, it reduces exactly to the group algebra $\mathbb{C}[\Gamma]$. For instance, the convolution algebra $\mathscr{C}_c(\mathbb{Z})$ is the space of complex valued $\mathbb{Z}$-sequences with Cauchy product.

The **integrated form** of a representation (π, V) of G is the map $\widehat{\pi} : \mathscr{C}_c(G) \to \mathscr{B}(V)$ from the convolution algebra to the space of bounded operators on V, defined as follows :

$$\widehat{\pi}(f)(v) := \int_G f(g)\pi(g)v\,dg. \tag{7}$$

The integrated form verifies the property that $\widehat{\pi}(f_1 \star f_2) = \widehat{\pi}(f_1) \cdot \widehat{\pi}(f_2)$ for all $f_1, f_2 \in G$. It equips the vector space V with a left-module structure over the algebra $\mathscr{C}_c(G)$ which depends only on the representation. The transformation $\pi \mapsto \widehat{\pi}$ even realizes the equivalence of categories of theorem 1.1.1.

When G is abelian, its **Pontryagin dual** is the space of its unitary characters $\widehat{G} := \mathrm{Hom}(G, U(1))$. It is a locally compact group with the property $\widehat{\widehat{G}} \simeq G$, known as Pontryagin self-duality. The Fourier transform sends any element of the convolution algebra of G to a function vanishing at infinity on its Pontryagin dual :

$$\widehat{\bullet} : \mathscr{C}_c(G) \longrightarrow \mathscr{C}_0(\widehat{G}).$$

In the abelian case, it constructs a bridge between representation theory and classical geometry. As the convolution algebra is *commutative* if and only if the underlying group is abelian, this Fourier-type argument doesn't hold in generality. It is because the expected *dual space* or *quantum space* is not necessarily geometric but of representation type.

1.1.3 Completions

1.1.3.1 $C^\star$-algebra

We say that a representation (π, V) of G is **unitary** if for every $g \in G$ the operator $\pi(g)$ is unitary as operator on V [Mac77]. Two unitary representations (π, V) and (τ, W) are said to be **equivalent**, and we note $V \sim W$, if it exists an operator $T : V \to W$ such that for all $g \in G$, $T\pi(g) = \tau(g)T$.

We define the **unitary dual** of a locally compact group G as the set of equivalence classes of irreducible unitary representations of G :

$$\widehat{G} := \{\text{unitary irreducible representations of } G\}/\sim.$$

It is a topological space equipped with the **Fell topology** [Fel62]. When G is abelian, irreducible representations are one-dimensional and the unitary dual is nothing else than the Pontryagin dual. For non-abelian group the unitary dual is more complicated and is a central motivation for the *Langlands program* [BSV24].

Definition 1.1.2. *The **group** $C^\star$**-algebra** $C^\star(G)$ of a locally compact group G is the completion of $\mathscr{C}_c(G)$ with respect to the* $\star$**-norm** *defined as :*

$$\|f\|_\star := \sup_{[\pi]\in\widehat{G}} \|\widehat{\pi}(f)\|.$$

This algebra plays a key role in representation theory and non-commutative geometry, see [Dix77]. It is the $C^\star$-algebra encoding all the unitary representations of G. The Fell topology is set to obtain a bijection between the unitary dual $\widehat{G}$ and the set of primitive ideals of $C^\star(G)$. It produces a topological approach to a representation theory problem. When G is abelian, the Fourier transform $f \leftrightarrow \widehat{f}$ even gives the identification

$$C^\star(G) \simeq \mathscr{C}_0(\widehat{G}). \tag{8}$$

1.1.3.2 Reduced $C^\star$-algebra

A **tempered representation** is a unitary irreducible representation which is weakly contained in the left-regular representation [Fel62]. The **tempered dual** of G is defined as

$$\widehat{G}_t := \{\text{tempered representations of } G\}/\sim.$$

It is a subspace of unitary dual and inherits naturally of the induced Fell topology. Now, the corresponding $C^\star$-algebra is known as the *reduced* $C^\star$-algebra of the group.

Definition 1.1.3. *The **reduced** $C^\star$**-algebra** $C^\star_r(G)$ of a locally compact group G is the completion of $\mathscr{C}_c(G)$ with respect to the **reduced** $\star$**-norm** defined from the integrated form of the left-regular representation λ :*

$$\|f\|_r := \|\widehat{\lambda}(f)\|.$$

When the group is compact (and more generally **amenable**), the Peter-Weyl theorem asserts that every irreducible unitary representation is tempered, which gives $C^\star(G) \simeq C^\star_r(G)$ in that case.

1.2 Crossed product algebras

Crossed product algebras extend the idea of group $C^\star$-algebras by allowing a group G to act on another algebra A. The construction $A \rtimes G$ encodes both the algebraic structure of A and the dynamics of the group action, generalizing the case $A = \mathbb{C}$ which corresponds to $C^\star(G)$. Concretely, elements are functions from G to A with convolution twisted by the action. Crossed products thus provide a noncommutative analogue of transformation group spaces and dynamical systems. They are central in noncommutative geometry : when $A = \mathscr{C}_0(X)$, the crossed product reflects the G-action on the topological space X. This section introduces algebraic crossed products, explains their functorial properties, and presents fundamental results such as Green's imprimitivity theorem. This part is inspired from the book [Wil07], which stands as the classical reference for the subject.

1.2.1 Algebraic crossed product

A $C^\star$**-dynamical system** is a triple (A, G, α) of a $C^\star$-algebra A, a locally compact group G and an action α of G on A by continuous automorphisms. Let $\mathscr{C}_c(G, A)$ be the space of compactly supported continuous functions. We define the twisted convolution by :

$$(f_1 \star f_2)(g) := \int_G f_1(s)\alpha_s(f_2(s^{-1}g))ds.$$

The algebra $(\mathscr{C}_c(G, A), \star)$ with this convolution product is called **algebraic crossed product** and denoted $A \rtimes G$.

This notion is a generalization of the group algebras we saw because when $A = \mathbb{C}$ is equipped with the trivial action, the dynamical system $(\mathbb{C}, G, id)$ induces nothing other than the usual convolution algebra $\mathscr{C}_c(G)$ of the group.

If X is a topological space endowed with an action α of G, we can extend the action on $A = \mathscr{C}_0(X)$ by the formula $(\alpha_g f)(x) := f(\alpha_g^{-1}x)$, making $(\mathscr{C}_0(X), G, \alpha)$ a $C^\star$-dynamical system. The associated algebraic crossed product $\mathscr{C}_0(X) \rtimes G$ encapsulates both the action of the group and the geometry of the underlying space.

1.2.2 Full and reduced crossed product

A **covariant representation** of the dynamical system (A, G, α) on a Hilbert space H is the data of $\pi : A \to B(H)$, a non-degenerate representation of A and $U : G \to U(H)$ a strongly continuous unitary representation of G which intertwine as :

$$U_g \pi(a) U_g^\star = \pi(\alpha_g(a)).$$

We call **integrated form** of a covariant representation (π, U), the operator $\pi \rtimes U : \mathscr{C}_c(G, A) \to B(H)$ defined as

$$(\pi \rtimes U)(f)(v) := \int_G \pi(f(g)) U_g(v) dg.$$

It is a generalization of the integration form (7) we defined for usual convolution algebras. As before, it builds a left-module over the algebra $\mathscr{C}_c(G, A)$ which depends only on (π, U).

Definition 1.2.1. *The **crossed product** $A \rtimes_\alpha G$ is the completion of the algebraic crossed product $A \rtimes G$ with respect to the norm*

$$\|f\|_{max} := \sup_{(\pi,U) \text{ covariant}} \|(\pi \rtimes U)(f)\|.$$

Of course, when $A = \mathbb{C}$ is endowed with the trivial action, $\mathbb{C} \rtimes_{id} G$ is nothing other than the group $C^\star$-algebra $C^\star(G)$. The crossed product algebra $A \rtimes_\alpha G$ satisfies the universal property that each covariant representation (π, U) integrates to a representation $\pi \rtimes U$ over it which is non-degenerate and, in the converse sense, that any non-degenerate representation over the crossed product algebra is unitary equivalent to the integrated form of a covariant representation. The crossed product algebra encodes both the algebraic structure of the algebra A and the dynamics of the action of G. It is best thought of as the *noncommutative quotient* of the system (A, G, α).

Let us build some examples.

- In the easiest case when the algebra $A = \mathbb{C}$ is chosen trivial, covariant representations − *i.e.* non-degenerate representations of $\mathbb{C} \rtimes_{id} G$− are just unitary representations of the group G.
- If X is a locally compact space endowed with an action α of G, any G-vector bundle $p : E \to X$ (see §2.4) induces a non-degenerate representation of the crossed product algebra $\mathscr{C}_0(X) \rtimes_\alpha G$. Indeed, setting $H = \mathscr{L}^2(X, E)$ we obtain two intertwining operators $\pi : \mathscr{C}_0(X) \to B(H)$ and $U : G \to U(H)$ defined as :

$$\pi(f)(\xi)(x) := f(x)\xi(x) \text{ and } U(g)(\xi)(x) = g \cdot (\xi(\alpha_g^{-1} x)).$$

  Not all the covariant representations arise from this geometric construction because we imposed H to be a certain space of square integrable functions over X.

Definition 1.2.2. *The **reduced crossed product** $A \rtimes_{\alpha,r} G$ is the completion of $\mathscr{C}_c(G, A)$ for the reduced norm*

$$\|f\|_r := \|(\pi_{reg} \rtimes \lambda)(f)\|.$$

where (π_{reg}, λ) is the regular covariant representation on the Hilbert space $H = \mathscr{L}^2(G)$ defined with $(\pi_{reg}(a))(f)(x) := \alpha_x^{-1}(a) f(x)$ and $(\lambda(g))(f)(x) = f(g^{-1}x)$.

Again, when $A = \mathbb{C}$ with trivial action we recover the reduced $C^\star$-algebra $C_r^\star(G)$ of the group. Moreover, if G is compact (and more generally amenable), the Peter-Weyl theorem still asserts that the reduced crossed product $A \rtimes_{\alpha,r} G$ and the full crossed product $A \rtimes_{\alpha} G$ are isomorphic.

1.2.3 Properties of crossed products

The main properties of crossed product algebras can be expressed in a categorical setting and require the notion of Morita equivalences. Two rings A and B are **Morita equivalent** if their category of modules A-mod and B-mod are equivalent; we will write $A \sim B$. For instance, any ring A is Morita equivalent with all of its matrices spaces $M_n(A)$.

If A is a $C^\star$-algebra with an action of G as before and H is a subgroup of G, the $C^\star$-algebra $\mathscr{C}_0(G/H, A)$ is diagonally endowed with an action of G. The associated crossed product algebra can be described as follows.

Theorem 1.2.3 (Green Imprimitivity)**.** *[Wil07][§4.21] If G acts on A with an action α and H is a closed subgroup of G, there is Morita equivalence :*

$$\mathscr{C}_0(G/H, A) \rtimes_{\alpha} G \sim A \rtimes_{\alpha_{|H}} H.$$

This fundamental theorem tells us that the study of the crossed product algebra by a subgroup relies heavily on the understanding of the associated homogeneous space. Without coefficients, it only remains the translation action of the group G over G/H and we get as in [Wil07][§4.29] and [Rae92] :

$$\mathscr{C}_0(G/H) \rtimes_{\mathrm{lt}} G \sim C^\star(H). \tag{9}$$

where the underscript lt means *left-translation* action.

Remark This statement provides a deep duality between inducing and restricting representations in the context of group actions and algebras. In other words, even though the crossed product $\mathscr{C}_0(G/H) \rtimes_{\mathrm{lt}} G$ may seem far more complicated, its representation theory and structure are fundamentally governed by the subgroup H. It is one of the motivation for the Connes-Kasparov theorem, following the ideas behind Mackey's theory of induced representations and serves as a noncommutative geometric version of those classical results.

Now for the geometric point of view, take X as a G-space for a certain continuous action α and build the crossed product algebra associated to the dynamical system $(\mathscr{C}_0(X), G, \alpha)$. The corresponding crossed product algebra encodes the orbit structure of the action as states the following theorem from [Com84] and [Gre77].

Theorem 1.2.4. *When the action of G on X is free and proper, we have the Morita equivalence :*

$$\mathscr{C}_0(X) \rtimes_{\alpha} G \sim \mathscr{C}_0(X/G).$$

1.2.4 Smooth crossed product algebra

Fix G to be a real reductive group acting on a locally convex complex algebra A. The setup that will interest us is the **smooth crossed product algebra** :

$$A \rtimes G = \left(\mathscr{C}_c^\infty(G, A), \star\right).$$

It captures both the differential geometry of A and the dynamics of the group action.

Since G is reductive, it possesses a maximal compact subgroup K. The homogeneous space G/K is a euclidian space, thus contractible. Topologically, G and K are then smoothly homotopic and one expect some link between their respective crossed product algebras. The following is a theorem of Nistor which states that their periodic cyclic homology, we will define in section §4, are isomorphic along every conjugacy class.

Theorem 1.2.5 (Nistor)**.** *For every conjugacy class $\langle x \rangle$ of G, there exists an isomorphism :*

$$HP_\bullet(A \rtimes G)_{\langle x \rangle} \simeq HP_{\bullet + \dim(G/K)}(A \rtimes K)_{\langle x \rangle}.$$

Unfortunately, this isomorphism is not computable explicitly because it is the composition of several coboundary maps. One cannot expect to glue it along the conjugacy classes of G because the proof uses the topology of the centralizer G_x which drastically differ depending on the conjugacy class. The aim of this thesis is to provide a global computable isomorphism.

2 K-theory

Sommaire

K-theory was born at the crossroads of topology and representation theory. Its origins can be traced back to the study of vector bundles over a locally compact space X. The idea of A. Grothendieck is that instead of considering the set of isomorphism classes of vector bundles as a monoid for the Withney sum, he considers their formal differences which forms an abelian group. M. Atiyah and F. Hirzebruch [AH12] chose this abelian group to be the topological K-theory of X. More generally, the K-theory of an algebra is the abelian group of formal difference of idempotent matrices over it.

The K-theory of suspensions of a locally compact space lead to higher K-groups of it. The combine datum of all these K-groups builds a ring under the cup-product. This ring possess some important properties as *Morita invariance*, *homotopy invariance* and an *excision property*. Also, the cup-product via the Bott element realizes an isomorphism between the K-groups of same parity, making the theory $\mathbb{Z}/2\mathbb{Z}$-graded. The main early notes references are at least [Ati19], [Bot69] and [Kar78].

The aim of the Baum-Connes conjecture is to describe analytically the K-theory group $K_0(C_r^\star(G))$ of a reduced $C^\star$-algebra of a locally compact group. Some introductions are given by [90], [RLL00], [HR01] and [Weg93].

2.1 K-theory

We take X to be a compact space. Let us consider the set $\mathrm{VB}(X)$ of isomorphism classes of complex vector bundles over X; it is a semi-group for the Whitney sum. We denote by

$$G(X) := \mathrm{Gr}(\mathrm{VB}(X))$$

the associated Grothendieck group. It corresponds to a quotient of $\mathrm{VB}(X)$ with respect to the relation $E \sim E' + E''$ if we have an exact sequence $0 \to E' \to E \to E'' \to 0$ of vector bundles over X. The group $G(X)$ is abelian and given by the formal differences $[E] - [E']$ of such vector bundles. If $f : X \to Y$ is a continuous map between compact spaces, the pull back of any vector bundle over Y along f is a vector bundle over X. The functor $X \mapsto G(X)$ is then a contravariant functor from the homotopy category of compact spaces to the category of abelian groups.

Take two compact spaces $Y \subseteq X$. We define the **relative K-theory** associated to the pair (X, Y) as the kernel of the induced map on their Grothendieck groups

$$K^0(X, Y) := \ker(G(X) \longrightarrow G(Y)).$$

One can thought about this group as the isomorphism classes of morphisms $\phi : E \to E'$ of vector bundles over X whose fibers $\phi_x : E_x \to E'_x$ are isomorphisms for every $x \in Y$.

Lemma 2.1.1. *When Y is non-empty, the projection $\pi : X \to X/Y$ contracting $Y \subseteq X$ to a point $\{y\} \in X/Y$ realizes the isomorphism $\pi : K^0(X/Y, \{y\}) \xrightarrow{\sim} K^0(X, Y)$. In particular, when $X = \mathbb{R}^n$ and $Y = \mathbb{R}^n - \{0\}$ we get $K^0(S^n, \{\infty\}) \xrightarrow{\sim} K^0(\mathbb{R}^n, \mathbb{R}^n - \{0\})$.*

Definition 2.1.2. *We define the **K-theory** of a locally compact space X to be the K-theory of its one-point compactification $X^+ = X \cup \{\infty\}$ relatively to the point at infinity, and its higher K-theory groups as the K-theory of its successive suspensions :*

$$K^0(X) := K^0(X^+, \{\infty\}) \text{ and } K^{-n}(X) := K^0(X \times \mathbb{R}^n).$$

The transformations $X \mapsto K^{-n}(X)$ define contravariant functors from the homotopy category of locally compact spaces to the category of abelian groups. We denote by $K(X) = \bigoplus_{n \geq 0} K^{-n}(X)$ the direct sum of all the K-groups of X.

The purpose of this definition of K-groups using one-point compactification instead of the naive Grothendieck group of vector bundles is to control of support at infinity. Indeed, a complex vector bundle over a locally compact space X defines a class in $K^0(X)$ when it is trivial out a compact subspace of X. Also, the choice of the euclidian suspensions $X \times \mathbb{R}^n$ instead of the classical suspensions $\Sigma^n X$ in the definition of higher K-groups are equivalent because the compactification of euclidian suspension $(X \times \mathbb{R}^n)^+$ is homotopy equivalent to the suspension of its compactification $\Sigma^n(X^+)$.

Take X and Y two locally compact spaces and n, m two integers. The canonical projections $X \times \mathbb{R}^n \xleftarrow{\pi_X} X \times Y \times \mathbb{R}^{n+m} \xrightarrow{\pi_Y} Y \times \mathbb{R}^m$ are both continuous. The **tensor product** of two given vector bundles $E \to X \times \mathbb{R}^n$ and $F \to Y \times \mathbb{R}^n$ is a vector bundle $E \boxtimes F := \pi_X^\star(E) \otimes \pi_Y^\star(F)$ over $X \times Y \times \mathbb{R}^{n+m}$. This product of vector spaces extends to the K-theory groups. It is called **cup-product** :

$$\begin{array}{rccc} - \cup - : & K^{-n}(X) \times K^{-m}(Y) & \longrightarrow & K^{-n-m}(X \times Y) \\ & ([E], [F]) & \longmapsto & [E \boxtimes F] \end{array} . \tag{10}$$

When $X = Y$, the pullback of $E \boxtimes F$ along the diagonal map $\Delta : X \to X \times X$ defines a vector bundle $\Delta^\star(E \boxtimes F)$ over $X \times \mathbb{R}^{n+m}$. The group $K(X)$ is a ring for the product $[E] \times [F] = \Delta^\star([E] \cup [F])$.

Fix $X = \mathbb{R}^2$ and the trivial vector bundles $E = E' = \mathbb{C}^2 \times \mathbb{R}^2$. Take the morphism $\beta : E \longrightarrow E'$ whose fiber at the point $(x_1, x_2) \in \mathbb{R}^2$ is given by the square matrix :

$$\beta_{(x_1,x_2)} := \frac{1}{1 + \|(x_1, x_2)\|^2} \begin{pmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix} \in M_2(\mathbb{C})$$

The fibers are isomorphisms for any point of $\mathbb{R}^2$ except at $(0,0) \in \mathbb{R}^2$. In other words, β defines a class in relative K-theory $\beta \in K^0(\mathbb{R}^2, \mathbb{R}^2 - \{(0,0)\}) \simeq K^0(S^2, \{\infty\}) \simeq K^0(\mathbb{R}^2)$ where the first isomorphism have been established in the lemma 2.1.1 and the second follows from definition of K-groups.

Definition 2.1.3. *The **Bott element** is the class $\beta \in K^0(\mathbb{R}^2)$ defined above.*

Theorem 2.1.4 (Bott periodicity)**.** *The cup-product by the Bott element β realizes the isomorphism :*

$$- \cup \beta : K^{-n}(X) \xrightarrow{\sim} K^{-n}(X \times \mathbb{R}^2) =: K^{-n-2}(X).$$

In other words, K-theory is a $\mathbb{Z}/2\mathbb{Z}$-graded theory and we will mainly be interested by the groups $K^0(X)$ and $K^1(X)$ which will be called respectively **even** and **odd K-groups**.

2.2 Topological K-theory

We fix A to be a Banach algebra. We call $\mathscr{P}(A)$ the **space of idempotent matrices** of finite size with coefficients in A. This space stands as a non-commutative analogue of the vector bundles over topological spaces. Indeed, if $A = \mathscr{C}_0(X)$ is the space of vanishing at infinity functions over a given locally compact space, the Serre-Swan theorem [Swa62] asserts that its idempotent matrices are in bijection with the space of vector bundles over the compactification X^+ :

$$VB(X^+) \longleftrightarrow \mathscr{P}(\mathscr{C}_0(X)),$$

where we identify a vector bundle with its sections vanishing at infinity. The space of idempotent matrices defines a semi-group for the diagonal block law.

Definition 2.2.1. *We define the **K-theory** of A to be the Grothendieck group of homotopy classes of idempotent matrices on A and of its suspensions :*

$$K_0(A) := \mathrm{Gr}(\mathscr{P}(A)) \;\; and \;\; K_n(A) := K_0(A \otimes \mathscr{C}_0(\mathbb{R}^n)).$$

The transformations $A \mapsto K_n(A)$ define contravariant functors from the category of algebras to the category of abelian groups. We denote by $K_\bullet(A)$ the direct sum of all the K-groups of A :

$$K_\bullet(A) = \bigoplus_{n \geq 0} K_n(A).$$

As before, as for the topological K-theory the algebraic K-group $K_0(A)$ is a Grothendieck group, it is given by the formal differences $[e]-[e']$ of idempotent matrices over A.

Theorem 2.2.2. *The K-theory groups verify the following properties for all $n \geq 0$:*
- *(Non-commutative topological K-theory [Fuk66]) For any locally compact space X, the groups $K^{-n}(X)$ and $K_n(\mathscr{C}_0(X))$ are isomorphic under the identification of a complex vector bundle with its sections vanishing at infinity.*
- *(Morita invariance) If A and B are two $C^\star$-algebras with Morita equivalent underlying rings, then $K_n(A)$ and $K_n(B)$ are isomorphic.*
- *(Homotopy invariance) If $\phi, \psi : A \to B$ are homotopic algebra homomorphism (i.e. it exists a path of homomorphisms $(f_t : A \to B)_{t \in [0,1]}$ joining ϕ with ψ depending continuously on the parameter), then $K_n(\phi) = K_n(\psi)$.*

There is also a notion of cup product at the level of algebraic K-theory. Take two Banach algebras A and B and n, m two integers. The Kronecker product of an idempotent matrix e over $A \otimes \mathscr{C}_0(\mathbb{R}^n)$ with an idempotent matrix f over $B \otimes \mathscr{C}_0(\mathbb{R}^m)$ is a matrix $e \otimes f$ over $A \otimes B \otimes \mathscr{C}_0(\mathbb{R}^{n+m})$ which is again idempotent. The Kronecker product then extends at the level of K-group, it is called **cup-product** :

$$- \cup - : \begin{array}{ccc} K_n(A) \times K_m(B) & \longrightarrow & K_{n+m}(A \otimes B) \\ ([e],[f]) & \longmapsto & [e \otimes f] \end{array} . \tag{11}$$

Theorem 2.2.3 (Algebraic Bott periodicity)**.** *For any Banach algebra A, the cup product by the Bott element $\beta \in K^0(\mathbb{R}^2) \simeq K^0(\mathscr{C}_0(\mathbb{R}^2))$ realizes an isomorphism in algebraic K-theory :*

$$- \cup \beta : K_n(A) \xrightarrow{\sim} K_n(A \otimes \mathscr{C}_0(\mathbb{R}^2)) =: K_{n+2}(A)$$

Remark The *algebraic* Bott element $\beta = [e]-[e'] \in K_0(\mathscr{C}_0(\mathbb{R}^2))$ defined in 2.1.3 is represented by a difference of idempotent matrices $e, e' \in M_2(\mathscr{C}_0(\mathbb{R}^2))$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, these matrices are defined by the following for any $z \in \mathbb{C}$:

$$e(z) = \frac{1}{1+|z|^2}\begin{pmatrix} 1 & z \\ \overline{z} & |z|^2 \end{pmatrix} \text{ and } e'(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We will mainly be interested by the two first algebraic K-groups $K_0(A)$ and $K_1(A)$. Despite the formal definition of $K_1(A)$ in terms of idempotent matrices over the first suspension algebra, the group possesses another equivalent description when A is unitary. Two invertible matrices of any size $u, v \in GL_\infty(A)$ are said to be homotopic, and we note $u \sim v$, if there exists a path connecting them up to top-left corner $GL_n(A) \subseteq GL_{n+1}(A)$ inclusions. The group $K_1(A)$ becomes the space of invertible matrices of any dimension over A modulo homotopy.

Let us compute some classical examples. When $A = \mathbb{C}$, $K_0(\mathbb{C})$ is the Grothendieck group of the space of idempotent matrices over $\mathbb{C}$. We can associate to any idempotent matrix its rank, which is an integer, and extend it up to an isomorphism $K_0(\mathbb{C}) \simeq \mathbb{Z}$.

Also, as all the spaces $GL_n(\mathbb{C})$ are connected, every invertible matrix define the same class in $K_1(\mathbb{C}) = 0$.
The case of compact operators over a infinite dimensional Hilbert space $A = \mathcal{K}(H)$ can be deduced from the fact that $\mathcal{K}(H)$ is Morita equivalent to $\mathbb{C}$, and then possesses the same K-theory groups $K_0(\mathcal{K}(H)) \simeq \mathbb{Z}$ and $K_1(\mathcal{K}(H)) = 0$ due to theorem 2.2.2.
Another way to compute K-theory group is provided by the excision property as follows.

Theorem 2.2.4 (Excision). *[HR01] From an extension of algebras $0 \to J \to A \xrightarrow{\pi} A/J \to 0$ there exists two maps $\partial_0 : K_1(A/J) \to K_0(J)$ and $\partial_1 : K_0(A/J) \to K_1(J)$ which make the following six-term sequence exact :*

$$\begin{tikzcd} K_0(J) \arrow[r] & K_0(A) \arrow[r] & K_0(A/J) \arrow[d, "\partial_1"] \\ K_1(A/J) \arrow[u, "\partial_0"] & K_1(A) \arrow[l] & K_1(J) \arrow[l] \end{tikzcd}$$

Here is a example of use of the excision property. The **Toeplitz algebra** is the universal $C^\star$-algebra generated by a partial isometry, it fits into the long exact sequence

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow \mathcal{T} \longrightarrow \mathcal{C}(S^1) \longrightarrow 0.$$

We computed before the K-theory groups of $\mathcal{K}(H)$ and we can check that $K_i(\mathcal{C}(S^1)) \simeq K^i(S^1) \simeq \mathbb{Z}$ for $i = 0, 1$. In this case, the boundary map ∂_0 is an isomorphism, which yields, by the excision theorem above, the descriptions $K_0(\mathcal{T}) = \mathbb{Z}$ and $K_1(\mathcal{T}) = 0$.

The fact that K-theory groups preserve homotopy equivalences, Morita equivalences and possess the excision property makes naturally the K-theory as a $\mathbb{Z}/2\mathbb{Z}$-graded theory [CQ95]. The parity isomorphism being given by the cup-product with respect to the Bott element as seen above.

2.2.1 The dual Dirac element

There exists an element generalizing the Bott element for other class of algebras than the functions vanishing at infinity.

Definition 2.2.5. *Given a real vector space V and a quadratic form $Q : V \longrightarrow \mathbb{C}$, the **Clifford algebra of V with respect to Q** is defined as :*

$$\mathrm{Cliff}(V, Q) := \widetilde{TV} / \langle v \otimes v - Q(v)1, v \in V \rangle$$

where $\widetilde{TV}$ denotes the tensor algebra of V with an adjointed unit.

The Clifford algebra verifies the property that, given a unital algebra A, any linear map $\Phi : V \to A$ verifying $\Phi(v)^2 = Q(v)1_A$ extends to a morphism of unital algebras

$$\mathrm{Cliff}(\Phi) : \mathrm{Cliff}(V, Q) \longrightarrow A. \tag{12}$$

The main example of Clifford algebra arises when V is endowed with a scalar product $\langle,\rangle$. In that case we write $\mathrm{Cliff}(V) = \mathrm{Cliff}(V, \|\cdot\|^2)$ and $\mathrm{Cliff}_{\mathbb{C}}(V)$ for its complexification. Write ε for the grading automorphism of $\mathrm{Cliff}(V)$ which is the identity on its even part and minus the identity on its odd part.

Lemma 2.2.6. *For any real vector space V, the following algebra homomorphism is stable under isometries and homotopic to the identity :*

$$\begin{array}{rccc} \Theta: & \mathscr{C}_c^\infty(V, \mathrm{Cliff}_{\mathbb{C}}(V)) \otimes \mathscr{C}_c^\infty(V, \mathrm{Cliff}_{\mathbb{C}}(V)) & \longrightarrow & \mathscr{C}_c^\infty(V, \mathrm{Cliff}_{\mathbb{C}}(V)) \otimes \mathscr{C}_c^\infty(V, \mathrm{Cliff}_{\mathbb{C}}(V)) \\ & a \otimes b & \longmapsto & (x,y) \mapsto b(x) \otimes \varepsilon(a(-y)) \end{array}.$$

Démonstration. Consider the description

$$\mathscr{C}_c^\infty(V, \mathrm{Cliff}_{\mathbb{C}}(V)) \otimes_\pi \mathscr{C}_c^\infty(V, \mathrm{Cliff}_{\mathbb{C}}(V)) \simeq \mathscr{C}_c^\infty(V \times V, \mathrm{Cliff}_{\mathbb{C}}(V \oplus V)).$$

The following smooth family of diffeomorphisms joins the map $(x, y) \mapsto (-y, x)$ to the identity

$$h_t : V \times V \longrightarrow V \times V,\ (v, v') \mapsto (\cos(t)v + \sin(t)v', -\sin(t)v + \cos(t)v'),\ 0 \le t \le \pi/2.$$

Also the smooth homotopy of linear isometries

$$h'_t : V \oplus V \longrightarrow V \oplus V,\ (v, v') \mapsto (\cos(t)v + \sin(t)v', -\sin(t)v + \cos(t)v'),\ 0 \le t \le \pi/2$$

induce a smooth homotopy of algebra automorphisms $\Phi_t : \mathrm{Cliff}_{\mathbb{C}}(V \oplus V) \longrightarrow \mathrm{Cliff}_{\mathbb{C}}(V \oplus V)$ such that $\Phi_0 = id$ and $\Phi_1(a \otimes b) = b \otimes \varepsilon(a)$. The composition $f \mapsto \Phi_t \circ \Theta(f) \circ h_t$, $0 \le t \le \frac{\pi}{2}$ provides the expected smooth homotopy. □

Let M be an n-dimensional Riemannian manifold. Through the isomorphism $T_x M \simeq T_x^\star M$, the cotangent spaces are equipped with a natural scalar product induced by the metric on the manifold. The associated Clifford algebra $\mathrm{Cliff}_{\mathbb{C}}(T^\star M)$ defines a vector bundle called **Clifford cotangent bundle** over M whose fiber at each point are $\mathrm{Cliff}_{\mathbb{C}}(T_x^\star M)$.

Definition 2.2.7. *The* ***smooth Clifford module*** *associated to the Riemannian manifold M is defined to be the space of smooth sections with compact support of the Clifford cotangent bundle*

$$\mathscr{C}_\tau^\infty(M) := \Gamma_c^\infty(M, \mathrm{Cliff}_{\mathbb{C}}(T^\star M)).$$

Its ***Clifford module*** *is the $C^\star$-algebraic closure of $\mathscr{C}_\tau^\infty(M)$, and can be identified with the space of sections vanishing at infinity of the Clifford cotangent bundle $\mathscr{C}_\tau(M) := \Gamma_0(M, \mathrm{Cliff}_{\mathbb{C}}(T^\star M))$.*

Suppose that $\exp : T_x M \longrightarrow M$ is a diffeomorphism for any $x \in M$ and fix $x \in M$. The Riemann connection allows to identify all tangent spaces $T_y^\star M$ with $T_x^\star M$ by parallel transport along the geodesics $\exp\big(\mathbb{R} \cdot \exp^{-1}(y)\big)$. This identifications are isometric and give rise to an isomorphism of locally convex algebras :

$$\mathscr{C}_\tau(M) \simeq \mathscr{C}_0(T_x^\star M, \mathrm{Cliff}_{\mathbb{C}}(T_x^\star M)).$$

Take $\nu:\mathbb{R}\longrightarrow\mathbb{R}$ a smooth function which is equal to one on $]-\infty,\frac{1}{2}]$ and equal to zero on $[1,\infty[$ and put $\chi(\xi)=\nu(\|\xi\|)$ for all $\xi\in T_x^\star M$. We define two smooth maps

$$e_0,e_1:T_x^\star M\longrightarrow M_2\big(\mathrm{Cliff}_{\mathbb{C}}(T_x^\star M)\big)$$

by the formulas

$$e_0(\xi)=\frac{1}{\chi(\xi)^2+\|\xi\|^2}\begin{pmatrix}\chi(\xi)^2 & \chi(\xi)\cdot c(\xi)\\ \chi(\xi)\cdot c(\xi) & \|\xi\|^2\end{pmatrix}\ \text{ and }\ e_1(\xi)=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

The endomorphisms $e_0(\xi)$ and $e_1(\xi)$ are idempotent and e_0-e_1 is compactly supported in the unit disk of $T_x^\star M$ due to the definition of ν, hence extends to a difference of idempotent matrices over $\mathscr{C}_\tau^\infty(M)\subseteq\mathscr{C}_\tau(M)$.

Proposition 2.2.8. *Suppose* $\exp:T_xM\longrightarrow M$ *is a diffeomorphism for any* $x\in M$*. Then for* $i=0,1$*, the following morphisms of algebras are stable under isometries and homotopy equivalent :*

$$(e_i\otimes id),\ (id\otimes e_i):M_2(\mathscr{C}_\tau^\infty(M))\longrightarrow M_2(\mathscr{C}_\tau^\infty(M))\otimes M_2(\mathscr{C}_\tau^\infty(M)).$$

Démonstration. A direct computation shows that for any $\xi\in T_x^\star M$ and $i=0,1$ we have $\varepsilon(e_i(-\xi))=e_i(\xi)$. In other words, $(e_i\otimes id)=\Theta\circ(id\otimes e_i)$ where Θ is the morphism of the lemma 2.2.6 applied to $V=T_x^\star M$. Since Θ is stable under isometries and homotopy equivalent to the identity, the assertion follows. □

Definition 2.2.9. *Suppose* $\exp:T_xM\longrightarrow M$ *is a diffeomorphism for any* $x\in M$*. The* ***generalized Bott element*** *or* ***dual Dirac element*** *is the class*

$$\beta=[e_0]-[e_1]\in K_0(\mathscr{C}_\tau(M))$$

represented by the difference of the idempotent matrices e_0 *and* e_1 *over* $\mathscr{C}_\tau(M)$ *defined above.*

If $M=\mathbb{R}^2$ the algebra $\mathscr{C}_\tau(\mathbb{R}^2)$ is Morita equivalent to the algebra $\mathscr{C}_0(\mathbb{R}^2)$ and the dual Dirac element coincide with the Bott element.

2.3 K-theory of reduced group $C^\star$-algebras

Now, we fix G to be a locally compact group and $A=C_r^\star(G)$ be its reduced group $C^\star$-algebra. The computation of the K-groups of $C_r^\star(G)$ is one of the key motivation the Baum-Connes conjecture [PV81]. Partial answers have been provided from different frameworks and for different cases.

- When G is trivial, $C_r^\star(G)\simeq\mathbb{C}$ and we know $K_0(\mathbb{C})\simeq\mathbb{Z}$ and $K_1(\mathbb{C})=0$;
- When $G=\mathbb{Z}/2\mathbb{Z}$, the reduced $C^\star$-algebra $C_r^\star(\mathbb{Z}/2\mathbb{Z})$ is nothing other that the space of vanishing at infinity function over the Pontryagin duals of $\mathbb{Z}/2\mathbb{Z}$, which is $\mathbb{Z}/2\mathbb{Z}$ itself : $C_r^\star(\mathbb{Z}/2\mathbb{Z})\simeq\mathscr{C}_0(\mathbb{Z}/2\mathbb{Z})\simeq\mathbb{C}\oplus\mathbb{C}$. The K-theory of such a $C^\star$-algebra is $K_0(C_r^\star(\mathbb{Z}/2\mathbb{Z}))\simeq K_0(\mathbb{C})\oplus K_0(\mathbb{C})\simeq\mathbb{Z}\oplus\mathbb{Z}$, while $K_1(C_r^\star(\mathbb{Z}/2\mathbb{Z}))=0$.

- More generally when G is finite, we have $K_0(C_r^\star(G)) \simeq \mathbb{Z}^m$ and $K_1(C_r^\star(G)) = 0$, where m denotes the number of conjugacy classes of the group, which is also the number of equivalent classes of its unitary irreducible representations ;
- When G is abelian, we saw that $C_r^\star(G)$ is isomorphic to $\mathscr{C}_0(\widehat{G})$ via Fourier transform, which induces the descriptions $K_0(C^\star(G)) \simeq K^0(\widehat{G})$ and $K_1(C^\star(G)) \simeq K^1(\widehat{G})$;
- When G is compact, the Peter-Weyl theorem states that we can decompose the reduced $C^\star$-algebra of the group as a direct sum of matrix spaces [PW27] :

$$C_r^\star(G) \simeq \bigoplus_{[\pi]\in\widehat{G}} \mathscr{M}_{d_\pi}(\mathbb{C})$$

where d_π is the dimension of the representation π. As each summand $\mathscr{M}_{d_\pi}(\mathbb{C})$ is Morita equivalent to $\mathbb{C}$ whose K-groups are $K_0(\mathbb{C}) = \mathbb{Z}$ and $K_1(\mathbb{C}) = 0$, we get the following.

Theorem 2.3.1 (Peter-Weyl)**.** $K_0(C_r^\star(G)) \simeq \bigoplus_{[\pi]\in\widehat{G}} \mathbb{Z} \simeq R(G)$ *and* $K_1(C_r^\star(G)) = 0$.

2.4 Equivariant K-theory

We saw that K-theory provides a geometric tool via the study of vector bundles. It generalizes well to an equivariant setting. An introduction to G-equivariant K-theory is given by [Seg68]. We fix a compact Lie group G.

Let X be a compact space on which G acts continuously. A G**-equivariant vector bundle** (or G**-vector bundle**) is a complex vector bundle $p : E \to X$ such that the total space E is endowed with a continuous action of G, the map p is equivariant, and the translation on the fibers $g : E_x \to E_{\alpha_g x}$ is an isomorphism of complex vector spaces for all $g \in G$. The set of G-equivariant vector bundles $VB_G(X)$ is a semi-group with respect to the Whitney sum. We call G**-equivariant K-theory** of the space X, the Grothendieck group associated to it :

$$K_G^0(X) := \mathrm{Gr}(VB_G(X)).$$

It is an abelian group generated by the formal differences $[E]-[F]$ of G-vector bundles over X. As for K-theory, it defines a contravariant functor from the homotopy category of compact G-spaces to the category of abelian groups. We define the **higher equivariant K-theory** to be the equivariant K-groups of its successive suspensions :

$$K_G^{-n}(X) := K_G^0(\Sigma^n X).$$

When the group is trivial, we recover the classical K-theory $K^n(X)$ we defined in §2. Also, it is clear that the cup-product defined above preserves the different group action and then descends to equivariant K-theory. In particular, the euclidian space $\mathbb{R}^2$ is endowed with the trivial action of any compact Lie group G, thus the Bott element 2.1.3 again realizes a Bott periodicity.

Theorem 2.4.1 (Equivariant Bott periodicity)**.** *The Bott element $\beta \in K^0(\mathbb{R}^2)$ defined in 2.1.3 descends to a class in the G-equivariant K-theory of $\mathbb{R}^2$ for the trivial action. Its* *cup-product realizes for every compact Lie group G the isomorphism :*

$$\beta \cup - : K_G^{-n}(X) \xrightarrow{\sim} K_G^{-n-2}(X).$$

The equivariant K-theory is $\mathbb{Z}/2\mathbb{Z}$-graded and we will call **even** and **odd equivariant K-theory** the groups $K_G^0(X)$ and $K_G^1(X)$ respectively. One of the most important statement for equivariant K-theory is the Green-Julg theorem [EM09] which asserts that we can express equivariant K-theory in terms of crossed product algebras. The main statement happens in Kasparov KK-theory (see 6.1.1).

Theorem 2.4.2 (Green-Julg)**.** *When (A, K, α) is a $C^\star$-dynamical system with A unitary and K compact, we can compute equivariant K-theory using crossed product algebra :*

$$K_i^K(A) \simeq K_i(A \rtimes_\alpha K).$$

Proposition 2.4.3. *When G is a compact Lie group $R(G) \simeq K_G^0(pt)$.*

Démonstration. When G is compact, we already know that the representation ring $R(G)$ is isomorphic to $K_0(C^\star(G))$ due to theorem 2.3.1. But the $C^\star$-algebra $C^\star(G)$ is nothing other than the crossed product associated to the trivial dynamical system $(\mathbb{C}, G, id)$, thus $K_0(C^\star(G)) \simeq K_0(\mathbb{C} \rtimes_{id} G)$. Since the algebra $\mathbb{C}$ can be viewed as the set of functions over a point, the Green-Julg theorem 2.4.2 finally gives $K_0(\mathbb{C} \rtimes_{id} G) \simeq K_G^0(pt)$. Composing the isomorphisms ends up the proof. □

Proposition 2.4.4. *When G is a compact Lie group and H a closed subgroup $R(H) \simeq K_G^0(G/H)$.*

Démonstration. As before, since H is compact $R(H) \simeq K_0(C^\star(H))$ by the Peter-Weyl theorem 2.3.1. Also, the Green's imprimitivity theorem without coefficients (9) yields the isomorphism $K_0(C^\star(H)) \simeq K_0(\mathscr{C}(G/H) \rtimes_{lt} G)$ where the subscript lt denotes the left translation action of G on G/H. Finally, the Green-Julg theorem 2.4.2 establishes that $K_0(\mathscr{C}(G/H) \rtimes_{lt} G) \simeq K_G^0(G/H)$. Composing all these identifications ends up the proof. □

The isomorphism $R(H) \simeq K_G^0(G/H)$ above associates to any irreducible representation $(\pi, V) \in R(H)$ the class in $[i_H^G(\pi, V)] \in K_G^0(G/H)$ represented by the homogeneous G-vector bundle over G/H :

$$i_H^G(\pi, V) : G \times_H V \longrightarrow G/H, \ (g, v) \longmapsto gH. \tag{13}$$

It can be thought as an *induction* functor since the section of the vector space we obtain correspond to the underlying space of the induced representation $\Gamma(i_H^G(\pi, V)) \simeq \mathrm{Ind}_H^G(V)$.

Proposition 2.4.5. *When a compact group G acts freely and properly on a compact space X via a continuous action $K_G^i(X) \simeq K^i(X/G)$.*

Démonstration. We write α for the continuous action of G on X. The theorem 2.4.2 enables us to write the isomorphism $K_G^i(X) \simeq K_i(\mathscr{C}(X) \rtimes_\alpha G)$. Now, due to the Morita equivalence (theorem 1.2.4) yields $K_i(\mathscr{C}(X) \rtimes_\alpha G) \simeq K_i(\mathscr{C}(X/G))$ because the action is free and proper. The orbit space X/G is again compact, hence we can express the algebraic K-theory of its algebra of continuous functions as its topological K-theory : $K_i(\mathscr{C}(X/G)) \simeq K^i(X/G)$. Composing these identifications, yields the expected isomorphism. □

The equivariant theory provides a tool to compute homological invariants of geometric crossed product algebras, lying between representation theory and classical geometry.

3 Index theory and K-homology

Sommaire

This chapter establishes the analytic tools needed to connect geometry and operator algebras. Bott showed in [Bot65] that if K is a compact group, every irreducible representation, viewed as a generator of the representation ring $\operatorname{Rep}(K)$, is the index of a Dirac operator. Atiyah and Schmid [AS77] showed that every discrete series representation of a connected semisimple group with finite center is also the index of a Dirac operator. In other words, these operators play a central role in this theory. They provide canonical representatives of K-homology and their product realize interesting isomorphisms.

3.1 Index theory

3.1.1 The analytic index

We fix H to be a Hilbert space. A bounded linear operator over H is said to be **compact** if is it a limit of finite rank operators, and **Fredholm** if both its kernel and cokernel are finite dimensional. We write $\mathscr{L}(H)$ for the bounded operators over H, $\mathscr{K}(H)$ for the compact operators and $\mathscr{F}(H)$ for the Fredholm operators.

Definition 3.1.1. *If $F \in \mathscr{F}(H)$ is Fredholm, we call the following integer its* ***Fredholm index*** *(or* ***analytic index****) :*

$$\operatorname{Ind}(F) := \dim(\ker(T)) - \dim(\operatorname{coker}(T)) \in \mathbb{Z}.$$

Theorem 3.1.2 (Atkinson's)**.** *A bounded operator F is Fredholm if and only if it exists bounded operators S, R such that $SF-1$ and $FR-1$ are both compact. In other words, an operator is Fredholm if and only if it is invertible modulo compact operators.*

Intuitively, Fredholm operators are those operators that are invertible *if finite-dimensional effects are ignored.* They appear naturally in physics in equations involving elliptic operators.

3.1.2 Sobolev's spaces

The main examples of Fredholm operators arise from differential geometry. In order to introduce them, we need to recall some facts about Sobolev spaces. Fix $s \geq 0$.

Definition 3.1.3. *Define the s-**th Sobolev space** $\mathscr{H}_{(s)}(\mathbb{R}^n)$ associated to $\mathbb{R}^n$ as the completion of the algebra $\mathscr{C}_c^\infty(\mathbb{R}^n)$ of smooth functions with compact support over $\mathbb{R}^n$ with respect to the norm*

$$\|f\|_{(s)}^2 := \sum_{\substack{\alpha \in \mathbb{N}^n \\ |a| \leq s}} \|\partial^\alpha f\|^2_{L^2(\mathbb{R}^n)}.$$

It can be thought as the vector space of square-integrable functions on $\mathbb{R}^n$ whose all partial derivatives of order smaller than s are also square-integrable. We write $H_{(-s)}(\mathbb{R}^n)$ for the dual vector space of $H_{(s)}(\mathbb{R}^n)$.

When M is a compact manifold, there exists a finite partition of unity $\{\chi_i\}$ subordinate to a local coordinate chart, which is that any function $f \in \mathscr{C}_c^\infty(M)$ can be written as a sum $f = \sum_i \chi_i f$ where each $\chi_i f$ is a function on $\mathbb{R}^n$. Then the s-th Sobolev space of M is the vector space of square-integrable functions on M whose all partial derivatives relative this frame that are of order smaller than s are also square-integrable. In other words, $\mathscr{H}_{(s)}(M)$ is the completion of $\mathscr{C}_c^\infty(M)$ with respect to the norm

$$\|f\| := \sum_i \|\chi_i f\|_{(s)}.$$

This construction is independent of the choice of the subordinate partition of unity because two different choices of partition of unity define the same Hilbert space $\mathscr{H}_{(s)}(M)$. Moreover, if $M = \bigcup_{n \geq 0} K_n$ is the union of countably many increasing compact manifolds, its s-th Sobolev space is defined as the inductive limit of the Sobolev spaces associated to the compact submanifolds :

$$\mathscr{H}_{(s)}(M) := \varinjlim_n \mathscr{H}_{(s)}(K_n). \tag{14}$$

In other words, a square integrable function f defined on $M = \bigcup_{n \geq 0} K_n$ belongs to the Sobolev space $\mathscr{H}_{(s)}(M)$ if and only if it exists $n \geq 0$ such that the support of f lies in K_n and $f \in \mathscr{H}_{(s)}(K_n)$. In order to compare Sobolev spaces of different orders, we need to introduce the notion of p-summable operators.

Definition 3.1.4. *Fix a real number $p \geq 1$. A linear operator $T : \mathscr{H} \to \mathscr{H}'$ between two Hilbert spaces is called p-**summable** if it possesses a bounded p-th Schatten norm*

$$\|T\|_p := \mathrm{Tr}\left(\sqrt{T^\star T}^{\,p}\right)^{1/p}.$$

*The p-**th Schatten space** $\ell^p(\mathscr{H},\mathscr{H}')$ is the vector space of p-summable operators from $\mathscr{H}$ and $\mathscr{H}'$.*

If an operator $T:\mathscr{H}\to\mathscr{H}'$ is compact its square root operator $\sqrt{T^\star T}$ possesses a finite number of eigenvalues, hence compact operators are p-summable for any $p>1$. This statement realizes the p-th Schatten spaces as ideals $\ell^p(\mathscr{H},\mathscr{H}')\subseteq\mathscr{K}(\mathscr{H},\mathscr{H}')$ of the space of compact operators for any $p>1$. These ideals are important because they refine the notion of compactness of an operator. The following theorem states that the inclusion of a Sobolev into its predecessor is bounded for a great enough Schatten norm, and in particular compact.

Theorem 3.1.5 (Rellich's lemma)**.** *Let M be a compact Riemann-manifold. The canonical inclusion*

$$\iota:\mathscr{H}_{(s)}(M)\hookrightarrow\mathscr{H}_{(s-1)}(M)$$

is bounded for the p-Schatten norm when $p>\dim(M)$, i.e. $\iota\in\ell^p\left(\mathscr{H}_{(s)}(M),\mathscr{H}_{(s-1)}(M)\right)$.

3.1.3 Elliptic operators

Consider a non-necessarily compact manifold M and a vector bundle $E\to M$ of rank n over it. Around any point $x\in M$, it exists a local coordinate system $(x_1,\cdots,x_n)$ and a local frame $(s_1,\cdots,s_n)$ such that any section $s\in\Gamma^\infty(M,E)$ can be written as $s(x)=\sum_{j=1}^n f_j(x_j)s_j(x_j)$ for some $f_j\in\mathscr{C}^\infty(M)$.

Definition 3.1.6. *A **differential operator** of order m acting on the smooth sections of $E\to M$ is a linear map $D:\Gamma^\infty(M,E)\longrightarrow\Gamma^\infty(M,E)$ which, in local coordinate $(x_1,\cdots,x_n)$ and in local frame $(s_1,\cdots,s_n)$ around a point $x\in M$, can be written as a sum of partial derivatives :*

$$(Ds)_i(x)=\sum_{|\alpha|\le m}\sum_{j=1}^n g_{i,j}^\alpha(x)\frac{\partial^{|\alpha|}f_j}{\partial^{\alpha_1}x_1\cdots\partial^{\alpha_n}x_n}s_j$$

where $s(x)=\sum_{j=1}^n f_j(x_j)s_j(x_j)\in\Gamma^\infty(M,E)$ and $g_{i,j}^\alpha\in\mathscr{C}^\infty(M)$.

If $f\in\mathscr{H}_{(1)}(M)$, then by definition $df\in\mathscr{L}^2(M)=\mathscr{H}_{(0)}(M)$. In other words, the product by the differential operator d of order 1 decreases the order of the Sobolev space by 1. More generally, if $f\in\mathscr{H}_{(s)}(M)$ and D is differential operator of order m, then $Df\in\mathscr{H}_{(s-m)}(M)$. The Sobolev's lemma states that this multiplication by a differential operator is always bounded.

Theorem 3.1.7 (Sobolev's lemma)**.** *Let M be a Riemann-manifold and D be a differential operator of order m. Then the multiplication with D is bounded :*

$$\mathscr{H}_{(s)}(M)\xrightarrow{D}\mathscr{H}_{(s-m)}(M).$$

Let D is a differential operator acting on a vector bundle $E\to M$. Denote by $\rho(f)$ the pointwise multiplication by a smooth function $f\in\mathscr{C}_c^\infty(M)$ on $\Gamma^\infty(M,E)$. It is natural to consider the *multiplication* operator $[D,\rho(f)] : \Gamma^\infty(M,E) \longrightarrow \Gamma^\infty(M,E)$. When f is compactly supported, this operator is bounded [HR01][p.274]. The **principal symbol** of D is the linear map $\sigma_D : T^\star M \longrightarrow End(E)$ defined as the highest homogeneous component of D, which is the action of $T^\star M$ on E making any cotangent vector of the form $(x, df) \in T^\star_x M$ act pointwise via the associated multiplication operator

$$\sigma_D(x,df)\big(s(x)\big) = ([D,\rho(f)]s)(x)$$

for $x \in M$, $f \in \mathscr{C}^\infty(M)$ and $s \in \Gamma^\infty(M,E)$.

Definition 3.1.8. *An* ***elliptic operator*** *is a differential operator P whose symbol is invertible. It is equivalent to ask P to be invertible modulo lower order operators, i.e. that it exists S and S' two operators such that the order of $SP-1$ and $TS'-1$ are strictly smaller than the order of P.*

Theorem 3.1.9. *An elliptic operator P over a compact manifold is always Fredholm. Furthermore*

$$\ker(P) \subseteq \mathscr{L}^2(M,E) \ \text{ and } \ \operatorname{coker}(P) \subseteq \mathscr{L}^2(M,E)$$

consist only on smooth sections of E.

This theorem follows from Rellich's and Sobolev's arguments. It relates the study of elliptic operators over a manifold with the comprehension of the Fredholm operators over the Hilbert space of its square-integrable functions. When the manifold is not compact, its elliptic operators aren't necessarily Fredholm. In 1963, M. Atiyah and I. Stinger proved that the analytic index of an elliptic operator coincide with its *topological index* defined using differential geometric tools such as Todd and Chern classes [AS63].

3.2 K-homology

Definition 3.2.1. *A* ***Fredholm module*** *over a complete locally convex algebra A is a pair (ρ,F) where $\rho : A \to \mathscr{L}(H)$ is a bounded representation, and F a bounded self-adjoint operator on H verifying*

$$\rho(a)(F^2-1) \in \mathscr{K}(H) \text{ and } \rho(a)F - F\rho(a) \in \mathscr{K}(H) \text{ for all } a \in A.$$

We will be mainly interested on the cases where A is a space of functions on an open manifold.

The terminology *Fredholm* module arises from the fact that when $A = \mathbb{C}$ or the underlying space is a point, Fredholm modules are exactly given by pairs where F is a Fredholm operator. In other words, Fredholm modules stand as a generalization of the usual notion of Fredholmness. Sometimes, one ask the operators $\rho(a)(F^2-1)$ and $\rho(a)F - F\rho(a)$ to be not only compact but to lie in a certain operator ideal, for instance the p-th Schatten ideal $\ell^p(H)$ defined in 3.1.4. In that case, we talk about *p-summable*

Fredholm modules.

A Fredholm module (ρ, F) over a $\mathbb{Z}/2\mathbb{Z}$-graded algebra is called **even** if the underlying Hilbert space decomposes as $H = H_+ \oplus H_-$, ρ is degree preserving and F is odd. It is said to be **odd** is H is ungraded.

Example(s) - Consider the algebra $A = \mathscr{C}(S^1)$ of continuous functions on the circle. The Hilbert space $H = \mathscr{L}^2(S^1)$ is naturally equipped with the bounded representation ρ of $\mathscr{C}(S^1)$ given by the pointwise multiplication. We define the operator F on the trigonometric basis $(e^{2i\pi n\theta})_{n\in\mathbb{Z}}$ to be $F(e^{2i\pi n\theta}) = |n| \cdot e^{2i\pi n\theta}$. It may be thought as the phase of the differential operator $D = -i\frac{d}{d\theta}$. The pair (ρ, F) is an odd Fredholm module of $\mathscr{C}(S^1)$.

- Take M to be a non-necessarily compact manifold and P a self-adjoint elliptic operator of order 1 over M associated to a vector bundle $E \to M$. The Hilbert space of square-integrable sections $H = \mathscr{L}^2(M, E)$ inherits from the $\star$-representation ρ of $\mathscr{C}_0(M)$ given by pointwise multiplication. Consider the operator $F = P(1+P^2)^{-1/2} \in \mathscr{L}(H)$. For all $f \in \mathscr{C}_0(M)$, $\rho(f)(F^2-1)$ is compact as product of compact operators (f vanishes at infinity) and $\rho(f)F - F\rho(f)$ is compact as a pseudo-differential elliptic operator of order -1. The pair

$$\left(\rho = \text{pointwise mult.}, F = \frac{P}{\sqrt{1+P^2}}\right) \tag{15}$$

thus defines an odd Fredholm module over $\mathscr{C}_0(M)$.

Two Fredholm modules (ρ, F) and (ρ', F') are said to be **operator homotopic** if $\rho = \rho'$ and there exists a norm continuous path $(F_t)_{t\in[0,1]}$ connecting F to F' such that (ρ, F_t) is a Fredholm module for all $t \in [0,1]$. If the commutator $\rho(a)F - F\rho(a) = 0$ and $\rho(a)F^2 = \rho(a)$ for all $a \in A$, we say that that the Fredholm module (ρ, F) is **degenerate**. Finally, (ρ, F) and (ρ', F') are **equivalent** if there exists two degenerated Fredholm modules (ρ_0, F_0) and (ρ_1, F_1) such that $(\rho, F) \oplus (\rho_0, F_0)$ is operator homotopic to $(\rho', F') \oplus (\rho_1, F_1)$.

Definition 3.2.2. *The **K-homology groups** of a $C^\star$-algebra A are the abelian groups*

$$K^0(A) \ \textit{and} \ K^1(A)$$

*of equivalence classes of Fredholm modules over A that are even and odd respectively. When $A = \mathscr{C}_0(X)$ is the algebra of functions vanishing at infinity over a locally compact space X, we denote the **K-homology** of X as*

$$K_0(X) = K^0(\mathscr{C}_0(X)) \ \textit{and} \ K_1(X) = K^1(\mathscr{C}_0(X)).$$

The K-homology groups of a manifold can be described via the elliptic operators over it. It is a theorem of M. Atiyah, I. Singer and G. Kasparov.

Theorem 3.2.3. *[Kas75] Let M be a second countable manifold. The set of homotopy* *classes of elliptic operators on M generates the K-homology of M via the construction of (15) :*

$$\mathrm{Ell}(M)_{/\text{homotopy}} \twoheadrightarrow K_\bullet(M).$$

It is a bijection at the level of properly supported and essentially self-adjoined elliptic operators.

The transformations $X \mapsto K_m(X)$ define for $m = 0, 1$ covariant functors from the homotopy category of locally compact spaces to the category of abelian groups. However, the direct sum $K_\bullet(X) = K_0(X) \oplus K_1(X)$ doesn't inherit of a ring structure contrary to $K^\bullet(X)$ but is a module over it. Indeed, given classes $[E] \in K^{-n}(X)$ (*i.e.* a vector bundle $E \to X \times \mathbb{R}^n$ trivialized around the infinity) and $[P] \in K_m(X)$ (*i.e.* an elliptic operator over X associated to a vector bundle $S \to X$), one can choose a connection on E and build the twisted elliptic operator P_E on X whose principal symbol is $\sigma(P_E) = \sigma(P) \otimes id_E$ acting on $S \otimes E$. It represents a class $K_{n+m}(X)$ that depends only on E and P modulo homotopy. It defines a linear map called **cap-product** :

$$\begin{array}{rccc} -\cap - : & K^{-n}(X) \times K_m(X) & \longrightarrow & K_{n+m}(X) \\ & ([E],[P]) & \longmapsto & [E] \cap [P] := [P_E] \end{array}.$$

The cap-product and cup-product are related by the identity $\psi(\sigma \cap \phi) = (\phi \cup \psi)(\sigma)$, making the K-homology vector space $K_\bullet(X)$ a module over the K-theory ring $K^\bullet(X)$.

3.3 Equivariant K-homology

Let X be a locally compact space, endowed now with a proper action of a second countable locally compact group G. This section proposes an analytic model for equivariant K-homology which is inspired from the references [Val02] and [HR01].

Definition 3.3.1. *A G-**equivariant Fredholm module** over a complete locally convex G-algebra A is a triplet (ρ, F, U) where (ρ, F) is a Fredholm module over A on some Hilbert space H, and $U : G \to U(H)$ is a unitary representation such that for any $g \in G$ and $a \in A$*

$$U_g F = F U_g \ \text{ and } \ U_g \rho(a) U_g^{-1} = \rho(g \cdot a).$$

A G-equivariant Fredholm module (ρ, F, U) is **even** if (ρ, F) itself even and the unitary representation of G is degree preserving. The G-equivariant Fredholm module is said to be **odd** when H is ungraded.

Two G-equivariant Fredholm modules (ρ, F, U) and (ρ', F', U') are said to be **operator homotopic** if $\rho = \rho'$, $U = U'$ and if there exists a norm continuous path $(F_t)_{t\in[0,1]}$ connecting F to F' such that for each $t \in [0,1]$ and $g \in G$, $U_g F_t = F_t U_g$. Equivalence of G-equivariant Fredholm modules is analogous to the one of classical Fredhom modules according to this notion of homotopy.

Definition 3.3.2. *We call G-**equivariant K-homology** of a complete locally convex algebra A, the abelian groups*

$$K^0_G(A) \ \text{and} \ K^1_G(A)$$

of equivalence classes of G-equivariant Fredholm module over A that are even and odd respectively. When $A = \mathscr{C}_0(X)$ is the algebra of functions vanishing at infinity on a locally compact space X, we will write

$$K^G_0(X) = K^0_G(\mathscr{C}_0(X)) \ \text{and} \ K^G_1(X) = K^1_G(\mathscr{C}_0(X))$$

*for the G-**equivariant K-homology** of X. Finally, when $G = \{1\}$ is the trivial group, we recover the K-homology $K^m_{\{1\}}(A) = K^m(A)$ of the algebra.*

As for K-homology, its equivariant version defines a covariant functor $X \mapsto K^G_m(X)$ for $m = 0, 1$ from the homotopy category of locally compact G-spaces to the category of abelian groups. The cap-product descends at the level of G-equivariant K-homology and G-equivariant K-theory, which is that the following linear map is well-defined :

$$- \cap - : K^{-n}_G(X) \times K^G_m(X) \longrightarrow K^G_{n+m}(X),$$

making $K^G_\bullet(X)$ as a module over the ring $K^\bullet_G(X)$. As for the Poincaré duality, the cap-product via a well-chosen class might realize a link between these two *dual* theories. This class is the G-equivariant Dirac element.

3.3.1 The Dirac element

Fix a complete Riemannian-manifold M and a locally compact group G acting smoothly on M preserving the metric. We define on the space of compactly supported differential forms $\Omega^\star_c(M)$ the norm given by the integration with respect to the Riemannian volume form of M

$$\|\omega\|^2 := \int_M \|\omega_x\|^2 d\mathrm{vol}_x.$$

We write $\mathscr{H} = \mathscr{L}^2(\Omega^\star_c(M))$ for the completion of $\Omega^\star_c(M)$ with respect to this norm. The group G acts isometrically on it because the fixed metric on M is G-invariant. The Hodge operator

$$d + d^\star : \Omega^\star_c(M) \longrightarrow \Omega^\star_c(M)$$

is a differential operator acting on the sections of the vector bundle $E = \bigwedge T^\star M$. The square of its symbol $\sigma_{d+d^\star} : T^\star M \to End\left(\bigwedge T^\star M\right)$ is equal to the symbol of the De Rham-Laplace operator $\Delta = (d + d^\star)^2 = dd^\star + d^\star d$, which is $\sigma_{d+d^\star}(x,\xi)(\omega)^2 = \sigma_\Delta(x,\xi)(\omega) = -\|\xi\|^2 \cdot \omega$. By universal property of the Clifford algebra (12), the symbol $\sigma_{d+d^\star}$ extends to a morphism of algebras we denote $\Sigma : \mathrm{Cliff}(T^\star M) \longrightarrow End\left(\bigwedge T^\star M\right)$. Passing to the sections, it induces an action of the algebra of Clifford bundle sections vanishing at infinity $\mathscr{C}_\tau(M) = \Gamma_0(M, \mathrm{Cliff}(T^\star M))$ (see definition 2.2.7) on the differential forms $\Omega_c^\star(M)$. This action, called **Clifford multiplication**, extends as a representation of $\mathscr{C}_\tau(M)$ on $\mathscr{L}^2(\Omega_c^\star(M))$:

$$\rho_{d+d^\star} : \mathscr{C}_\tau(M) \longrightarrow End(\mathscr{L}^2(\Omega_c^\star(M))). \tag{16}$$

Since the complex function $f(z) = z(1+z^2)^{-1/2}$ is holomorphic and bounded on an horizontal strip near to the real line $\mathbb{R} \subset \mathbb{C}$, the linear operator

$$F := (d + d^\star)(1+\Delta)^{-1/2} : \mathscr{L}^2(\Omega_c^\star(M)) \longrightarrow \mathscr{L}^2(\Omega_c^\star(M)) \tag{17}$$

is bounded and pseudo-differential of order 0. Finally, since G preserves the metric of the manifold, it is clear that the induced action

$$U : G \to End(\mathscr{L}^2(\Omega_c^\star(M))) \tag{18}$$

verifies $U_g F = F U_g^{-1}$ and $U_g \rho(s) U_g = \rho(g \cdot s)$ for any $g \in G$ and $s \in \mathscr{C}_\tau(M)$. If the manifold is even dimensional, $E = \bigwedge T^\star M$ splits into even and odd parts, the representation $\rho_{d+d^\star}$ and the unitary action U are degree preserving while the operator bounded F is odd. When M is odd-dimensional, the vector bundle E doesn't split.

Theorem 3.3.3. *If a locally compact group G acts smoothly on a complete Riemannian-manifold M preserving the metric, the following tuple is a G-equivariant Fredholm module over the Clifford algebra $\mathscr{C}_\tau(M)$*

$$\alpha := \left(\mathscr{H} = \mathscr{L}^2(\Omega_c^\star(M)),\ \rho_{d+d^\star},\ F = \frac{d+d^\star}{(1+\Delta)^{1/2}},\ U \right).$$

In particular, it represents a class $\alpha \in K^G_{\dim(M)}(M) = K_G^{\dim(M)}(\mathscr{C}_\tau(M))$ that we call ***Dirac element****.*

Theorem 3.3.4 (Kasparov-Poincaré duality)**.** *The product by the Dirac element $\alpha \in K^G_{\dim(M)}(M)$ realizes the following isomorphism of abelian groups :*

$$- \cap \alpha : K_G^{-n}(M) \xrightarrow{\sim} K^G_{n+\dim(M)}(M).$$

4 Cyclic homology

Sommaire

Over the past few decades, the language of homological algebra has increasingly become central to the study of algebraic and geometric structures, particularly in contexts where traditional tools of commutative geometry are no longer available. One of the most fruitful developments in this direction has been the emergence of cyclic homology, a theory that arose at the intersection of algebra, topology, and analysis, and has proven to be a key component in the framework of noncommutative geometry.

The origins of cyclic homology lie in the earlier theory of Hochschild homology, introduced to capture the failure of commutativity in associative algebras. Beyond this formal role, Hochschild homology gained a deeper geometric interpretation thanks to the Hochschild-Kostant-Rosenberg theorem, which identifies the Hochschild homology of a smooth commutative algebra with the space of differential forms. This result revealed that the Hochschild complex could serve as a noncommutative model for differential geometry, laying the groundwork for algebraic approaches to smooth manifolds.

Cyclic homology, initially defined by A. Connes in the early 1980's, defines a refinement of the Hochschild homology. It introduces additional structure into the Hochschild complex, either via the *cyclic operator* or through the bicomplex constructed with the *boundary operator* introduced earlier by Rinehart. A particularly compelling aspect of cyclic homology is its interaction with K-theory. While algebraic K-groups are notoriously difficult to compute, they admit a canonical map, the *Chern character*, into the periodic cyclic homology which satisfies crucial properties such as *Morita invariance*, *excision*, and *homotopy invariance*. This map preserves much of the rich structure of the K-theory and facilitates computations by passing into a more accessible homological setting. In this sense, cyclic homology serves not only as a tool of intrinsic interest, but also as a crucial bridge between non-commutative algebra and topology.

4.1 Hochschild (co)homology

The Hochschild homology arose naturally in the algebraic theory as a machine to compute the default of commutativity of an algebra, and more generally of bimodule over it. This homology defines a non-commutative generalization of the notion of differential forms. This analogy is known as the Hochschild-Kostant-Rosenberg theorem and propelled the domain of non-commutative Geometry in the early 60's. A complete introduction is given in [Lod92].

Given two Fréchet vector spaces E and F, their **projective tensor product** [Gro54] denoted $E \otimes_\pi F$ is given by the completion of $E \otimes_{\mathbb{C}} F$ with respect to the largest semi-norm such that $\|e \otimes f\| = \|e\| \cdot \|f\|$. Given two compact manifolds M and N, the projective tensor product is the tensor product which realizes the identity

$$\mathscr{C}^\infty(M) \otimes_\pi \mathscr{C}^\infty(N) \simeq \mathscr{C}^\infty(M \times N).$$

We fix A to be a complex Fréchet algebra. Let us consider $C_n(A) := A^{\otimes_\pi n} \oplus A^{\otimes_\pi n+1}$. It corresponds to the space of differential forms $\Omega^n A$ under the identifications

$$a_1 \otimes \cdots \otimes a_n \longleftrightarrow da_1 \cdots da_n \qquad a_0 \otimes \cdots \otimes a_n \longleftrightarrow a_0 da_1 \cdots da_n.$$

We define the operator $b := \sum_{i=0}^n (-1)^i d_i : C_n(A) \to C_{n-1}(A)$ with

$$d_i(a_0 \otimes \cdots \otimes a_n) := \begin{cases} a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n & i = 0 \\ a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & 0 < i < n \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & i = n \end{cases} \tag{19}$$

This operator verifies the relation $b \circ b = 0$.

Definition 4.1.1. *The **Hochschild homology of** A is the homology of the complex $C_\star(A)$ equipped with the differential b :*

$$HH_\star(A) := H_\star(C_\star(A), b).$$

The Hochschild homology groups are natural and functorial for the variable. They compute the *default of commutativity* of an algebra. In small degree, $HH_0(A)$ corresponds to the quotient of A by the space of commutators $HH_0(A) \simeq A/[A, A]$. More generally, the Hochschild homology is the derived functor of the left-exact functor $A \mapsto A/[A, A] = A \otimes_{A^e} A$ where $A^e := A \otimes A^{op}$. In other words :

$$HH_\star(A) \simeq \operatorname{Tor}_\star^{A^e}(A, A).$$

This point of view allows us to compute this homology using different complexes.

Whenever $P_\star(A)$ is a projective resolution of A as a bimodule over itself, *i.e.* a left A^e-module, one can compute its Hochschild homology as $HH_\star(A) \simeq H_\star(A \otimes_{A^e} P_\star(A))$ and the result doesn't depend on the resolution. The canonical example of projective (free) resolution is the *bar complex* $C_n^{\text{bar}}(A) := A^{\otimes n+1}$ with differential map

$$b' := b + (-1)^{n+1} d_n : C_n^{\text{bar}}(A) \to C_{n-1}^{\text{bar}}(A). \tag{20}$$

When $A = \mathscr{C}^\infty(M)$ is the algebra of smooth functions over a compact manifold, the associated bar complex can be described in terms of differential forms on the manifold. This yields to the following.

Theorem 4.1.2. (Hochschild-Kostant-Rosenberg's theorem [HKR09])
Let M be a compact manifold. The Hochschild homology of $\mathscr{C}^\infty(M)$ is isomorphic to the graded-algebra of differential forms over M

$$HH_\star(\mathscr{C}^\infty(M)) \simeq \Omega^\star(M).$$

Furthermore, the isomorphism is given by the anti-symmetrization map

$$\epsilon : f_0 \otimes \cdots \otimes f_n \mapsto f_0 df_1 \cdots df_n. \tag{21}$$

The meaning of this result is that the functor of smooth functions from differentiable manifolds to commutative algebras doesn't loose any cohomological information. This is the main idea behind Non-Commutative Geometry and the earlier works of A. Connes and B. Tsygan.

Definition 4.1.3. *We call the **Hochschild cohomology of** A the homology of the complex $C^\star(A) = \mathrm{Hom}_{A^e}(C_\star^{bar}(A), A)$ equipped with the dual differential $b^\vee := (-1)^n(- \circ b')$: $C^n(A) \to C^{n+1}(A)$ (see (20)) :*

$$HH^\star(A) := H^\star(C^\star(A), b^\vee).$$

This cohomology is functorial and natural as the Hochschild homology. It also computes the *default of commutativity* as in small degree since $HH^0(A)$ coincide with ZA, the center of A. More generally, this cohomology is the derived functor of the right exact functor $A \mapsto ZA = Hom_{A^e}(A, A)$:

$$HH^\star(A) \simeq \mathrm{Ext}^\star_{A^e}(A, A).$$

Then for any projective resolution $P_\star(A)$ of A as a bimodule over itself, $HH^\star(A)$ can be computed as the cohomology groups of $\mathrm{Hom}_{A^e}(P_\star(A), A)$.

4.2 Cyclic (co)homology

In 1963, G. Rinehart proposed a way to compute cyclic homology as a generalization of the De Rham cohomology, fitting with the Hochschild-Kostant-Rosenberg theorem above. He defined an operator B on the Hochschild complex whose aim is to play the role of the De Rham differential in a non-commutative framework. The juxtaposition of the differentials b and B gives naturally rise to the study of a bicomplex called $\mathscr{B}_\star(A)$ whose total homology is called *cyclic homology* and computes a non-commutative De Rham cohomology.

We fix A to be a unital complex Fréchet algebra. We define the **cyclic operator** $t : C_n(A) \to C_n(A)$ and the **norm operator** $N : C_n(A) \to C_n(A)$ as :

$$\begin{array}{rcl} t(a_0 \otimes \cdots \otimes a_n) & := & (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1} \\ t(a_1 \otimes \cdots \otimes a_n) & := & (-1)^{n-1} 1_A \otimes a_n \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{array} \quad \text{and} \quad N := 1 + t + \cdots + t^n.$$

These operators verify the identities $bN = Nb'$ and $b(1-t) = (1-t)b'$, where b and b' are defined in (19) and (20). The following is then well-defined.

Definition 4.2.1. *We call* ***cyclic homology of*** A *and we write* $HC_\star(A)$ *for the homology of the total complex of :*

$$CL_\star(A) := \begin{tikzcd} {} \arrow[d, "b"'] & {} \arrow[d, "-b'"'] & {} \arrow[d, "b"'] & {} \arrow[d, "-b'"'] & \\ A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & {} \arrow[l, "N"'] \\ A^{\otimes 2} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & {} \arrow[l, "N"'] \\ A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & A \arrow[l, "1-t"'] & {} \arrow[l, "N"'] \end{tikzcd} .$$

As for the Hochschild homology, these groups are stable under algebraic transformations. Any algebra homomorphism $f : A \to A'$ induces a linear map $f_\star : HC_\star(A) \to HC_\star(A')$. In small degree, $HC_0(A)$ is the quotient of A by the image of $b : A^{\otimes 2} \to A$ and the image of $1 - t : A \to A$. The cyclic operator t is trivial in zeroth degree and then $HC_0(A) \simeq A/[A, A] \simeq HH_0(A)$. We define the **extra degeneracy** $s : C_n(A) \to C_{n+1}(A)$ via

$$s(a_0 \otimes \cdots \otimes a_n) := 1_A \otimes a_0 \otimes \cdots \otimes a_n.$$

This operator defines a contracting homotopy of $(C^{\mathrm{bar}}_\star(A), b')$, *i.e.* of the odd columns of $CL_\star(A)$, and leads to the definition of the *Connes' boundary map* [Lod92], §2 :

$$B = (1-t) \circ s \circ N : C_n(A) \longrightarrow C_{n+1}(A). \tag{22}$$

The cyclic bicomplex $CL_\star(A)$ is then isomorphic to the one generated by b and B :

$$\mathscr{B}_\star(A) := \begin{tikzcd} \arrow[d, "b"] & \arrow[d, "b"] & \arrow[d, "b"] & \\ C_2(A) \arrow[d, "b"] & C_1(A) \arrow[l, "B"'] \arrow[d, "b"] & C_0(A) \arrow[l, "B"'] & \arrow[l, "B"'] \\ C_1(A) \arrow[d, "b"] & C_0(A) \arrow[l, "B"'] & & \\ C_0(A) & & & \end{tikzcd}$$

Proposition 4.2.2. *For any unital Fréchet algebra A : $HC_\star(A) \simeq H_\star(\mathrm{Tot}\,\mathscr{B}_\star(A))$.*

Through the anti-symmetrization map ϵ defined in the (21), we know that the Hochschild homology of the algebra of functions $\mathscr{C}^\infty(M)$ corresponds to the space of differential forms $\Omega^\star(M)$. It turns out the operator B becomes a multiple of the exterior derivative on the De Rham complex, which is that the following diagram commutes :

$$\begin{tikzcd} C_{n-1}(\mathscr{C}^\infty(M)) \arrow[r, "B"] \arrow[d, "\epsilon"', "\sim"] & C_n(\mathscr{C}^\infty(M)) \arrow[d, "\sim"', "\epsilon"] \\ \Omega^{n-1}M \arrow[r, "n\cdot d"] & \Omega^n M \end{tikzcd}$$

The bicomplex $\mathscr{B}_\star(\mathscr{C}^\infty(M))$ then encapsulated both the datum of differentials forms and of the exterior derivative. The observation of G. Rinehart in the early 60's is that the homology of its total complex needs to be related to the De Rham cohomology, which is the following theorem.

Theorem 4.2.3. *If M is a compact manifold, the cyclic homology of its algebra of smooth functions $\mathscr{C}^\infty(M)$ can be described using De Rham cohomology*

$$HC_n(\mathscr{C}^\infty(M)) \simeq \Omega^n(M)/d(\Omega^{n-1}(M)) \oplus H^{n-2}_{dR}(M) \oplus H^{n-4}_{dR}(M) \oplus \cdots$$

All the operators d_i, t and s can be *dualized* in the sense that on the dual complex $C^\star(A) = \mathrm{Hom}_{A^e}(C^{\mathrm{bar}}_\star(A), A)$ defined in 4.1.3 we have $d_i^\vee := -\circ d_i$, $t^\vee := -\circ t$ and $s^\vee := -\circ s$. They define of course $b^\vee := \sum_{i=0}^n (-1)^i d_i^\vee$, $(b')^\vee := \sum_{i=0}^{n-1}(-1)^i d_i^\vee$ and $N^\vee := 1 + t^\vee + \cdots (t^\vee)^n$. It gives rise to the dual theory as follows.

Definition 4.2.4. *We call **cyclic cohomology of** A and write $HC^\star(A)$ for the cohomology* *of the total complex of :*

$$CL^\star(A) := \begin{tikzcd}
{} & {} & {} & {} & \\
C^2(A) \arrow[u, "b^\vee"] \arrow[r, "(1-t)^\vee"] & C^2(A) \arrow[u, "-(b')^\vee"] \arrow[r, "N^\vee"] & C^2(A) \arrow[u, "b^\vee"] \arrow[r, "(1-t)^\vee"] & C^2(A) \arrow[u, "-(b')^\vee"] \arrow[r, "N^\vee"] & {} \\
C^1(A) \arrow[u, "b^\vee"] \arrow[r, "(1-t)^\vee"] & C^1(A) \arrow[u, "-(b')^\vee"] \arrow[r, "N^\vee"] & C^1(A) \arrow[u, "b^\vee"] \arrow[r, "(1-t)^\vee"] & C^1(A) \arrow[u, "-(b')^\vee"] \arrow[r, "N^\vee"] & {} \\
C^0(A) \arrow[u, "b^\vee"] \arrow[r, "(1-t)^\vee"] & C^0(A) \arrow[u, "-(b')^\vee"] \arrow[r, "N^\vee"] & C^0(A) \arrow[u, "b^\vee"] \arrow[r, "(1-t)^\vee"] & C^0(A) \arrow[u, "-(b')^\vee"] \arrow[r, "N^\vee"] & {}
\end{tikzcd}.$$

In degree zero, $HC^0(A)$ is the intersection of the kernel of $b^\vee : C^0(A) \to C^1(A)$ given by $(b^\vee)(a_0)(a_1) = a_0 a_1 - a_1 a_0$ and of $(1-t)^\vee : C^0(A) \to C^0(A)$. The kernel of the first is given by the center of A, denoted ZA, and the second is the zero map, which gives $HC^0(A) \simeq ZA \simeq HH^0(A)$ as an analogue result for cyclic homology.

A way to compute cyclic homology is to used the Connes' exact sequence. It is a long exact sequence involving Hochschild homology groups and cyclic homology groups.The first two columns of the bicomplex $CL_\star(A)$, computing the cyclic homology, are isomorphic to the complex $C_\star(A)$, computing the Hochschild homology. The image of the embedding $I : C_\star(A) \to CL_\star(A)$ is the kernel of the two-degree shift $S : CL_\star(A) \to CL_\star(A)[-2]$. The map S called **periodicity map** plays an important role in the theory. The following sequence of complexes is then exact :

$$0 \longrightarrow C_\star(A) \xrightarrow{I} CL_\star(A) \xrightarrow{S} CL_\star(A)[2] \longrightarrow 0. \tag{23}$$

By an argument of a long exact sequence we obtain the following statement.

Proposition 4.2.5 (Connes' exact sequence)**.** *[Lod92] For every unital Fréchet algebra A, the following sequences are exact for all n :*

$$HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A),$$

$$HH^n(A) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S} HC^{n+1}(A) \xrightarrow{I} HH^{n+1}(A),$$

where B is the Connes' boundary map defined in (22).

4.3 Relation with K-theory

The candidate homologies for the study of K-groups need to verify the same properties as K-theory, which is *stable under Morita equivalences, stable homotopies* and *with excision property* (see theorem 2.2.2 and 2.2.4). The main issue is that any homology theory respecting these properties is $\mathbb{Z}/2\mathbb{Z}$-graded. Both Hochschild homology and cyclic homology are then irrelevant, but the so called *periodic cyclic homology* verifies the expected properties and defines a $\mathbb{Z}/2\mathbb{Z}$-graded version of these theories. This homology appears to be a generalization of the De Rham cohomology and then defines a great receptacle for the Chern character coming from K-groups.

4.3.1 Periodic cyclic homology

We fix A to be a complex Fréchet algebra as before. We can define the differentials $b : C_n(A) \to C_{n-1}(A)$ and $B : C_n(A) \to C_{n+1}(A)$ on the graded vector space $C_\star(A)$ as in (19) and (22). These operators verify the property :

$$b^2 = B^2 = (b+B)^2 = 0.$$

We splits the Hochschild complex $C_\star(A)$ into the *even* and *odd* parts :

$$C_{\text{even}}(A) := \prod_{n\geq 0} C_{2n}(A) \quad \text{and} \quad C_{\text{odd}}(A) := \prod_{n\geq 0} C_{2n+1}(A).$$

The operator $b+B$ sends any even (*resp.* odd) degree element to a pair of odd (*resp.* even) degree elements, and then defines a differential map on the following $\mathbb{Z}/2\mathbb{Z}$-graded complex.

Definition 4.3.1. *[CQ95] We call **periodic cyclic complex of** A the following complex :*

$$\widehat{CC}(A) := \quad C_{even}(A) \underset{b+B}{\overset{b+B}{\rightleftarrows}} C_{odd}(A)$$

This periodic cyclic complex has only two homology groups. From now we will use the subscript "$\bullet$" for any $\mathbb{Z}/2\mathbb{Z}$-theory in order to distinguish with the classical "$\star$" corresponding to $\mathbb{Z}$-graded theories. Also, we will always write the even part at the left, and the odd part at the right.

Definition 4.3.2. *We define the **periodic cyclic homology of** A as the homology of the complex $\widehat{CC}(A)$, which is :*

$$HP_\bullet(A) := H_\bullet(\widehat{CC}(A)).$$

The periodic cyclic homology possesses the following properties you can find in [Lod92] and [Pus98].

Theorem 4.3.3. *Fix A and B two Fréchet algebras.*

- (EXCISION PROPERTY) *Any exact sequence* $0 \to I \to A \xrightarrow{p} A/I \to 0$ *yields a chain-homotopy equivalence :*

$$\widehat{CC}(I) \xrightarrow{\sim} \ker\left(\widehat{CC}(A) \xrightarrow{p} \widehat{CC}(A/I)\right)$$

- (6-TERMS EXACT SEQUENCE) *Any exact sequence* $0 \to I \to A \xrightarrow{p} A/I \to 0$ *yields*

6-terms exact sequence :

$$\begin{tikzcd} HP_0(I) \arrow[r] & HP_0(A) \arrow[r] & HP_0(A/I) \arrow[d] \\ HP_1(A/I) \arrow[u] & HP_1(A) \arrow[l] & HP_1(I) \arrow[l] \end{tikzcd}$$

- (TENSOR PRODUCT STABILITY) *If A and B are unital, there exists a chain-homotopy equivalence*
$$\widehat{CC}(A\otimes_\pi B) \xrightarrow{\sim} \widehat{CC}(A)\widehat{\otimes}_\pi\widehat{CC}(B).$$
- (MORITA INVARIANCE) *If A and B are Morita equivalent algebras their associated periodic cyclic complexes are homotopy equivalent, leading to an isomorphism between their periodic cyclic homology groups :*
$$\widehat{CC}(A) \xrightarrow{\sim} \widehat{CC}(B) \;\; \textit{and} \;\; HP_\bullet(A) \simeq HP_\bullet(B).$$
- (SMOOTH HOMOTOPY INVARIANCE) *Two Fréchet algebras morphisms $\phi_0, \phi_1 : A \to B$* ***smoothly*** *homotopy equivalent, i.e. such that it exists a family of morphisms $\phi_t : A \to B$ depending smoothly on $t \in [0,1]$, induce the same linear map at the level of periodic cyclic homology :*
$$HP_\bullet(\phi_0) = HP_\bullet(\phi_1).$$

The periodic cyclic homology might be thought as a *completion* of cyclic homology groups along the periodicity map $S : CL_\star(A) \longrightarrow CL_\star(A)[-2]$ defined in (23). Indeed, when S is surjective on homology groups we get

$$HP_\bullet(A) \simeq \varprojlim HC_{\bullet+2n}(A) \;\; \text{and} \;\; HP^\bullet(A) \simeq \varinjlim HC^{\bullet+2n}(A). \tag{24}$$

The main idea behind the periodic cyclic homology is that this homology stands as a non-commutative analogue of De Rham cohomology in the following sense.

Theorem 4.3.4 (Non-commutative De Rham cohomology)**.** *If M is a compact manifold, the periodic cyclic homology of its algebra $\mathscr{C}^\infty(M)$ of complex smooth functions can be expressed via even-odd De Rham cohomology groups through the anti-symmetrization map defined in (21) :*

$$HP_0(\mathscr{C}^\infty(M)) \simeq H_{dR}^{\text{even}}(M) \quad \textit{and} \quad HP_1(\mathscr{C}^\infty(M)) \simeq H_{dR}^{\text{odd}}(M).$$

Periodic cyclic homology realizes a powerful geometrical invariant for commutative algebras which is related to the K-theory via the Chern character. To introduce this construction, we will describe the Cuntz-Quillen's construction of the X-complex.

4.3.2 Cuntz-Quillen's X-complex

The X-complex of Cuntz-Quillen provides a description of periodic cyclic homology as a *derived functor* in terms of quasi-free extension of an algebra. This is the construction we will use to define the Chern character with values in periodic cyclic homology [CQ95].

Definition 4.3.5. *We say that a locally convex algebra R is **quasi-free** if for any nilpotent extension*

$$0 \longrightarrow I \longrightarrow S \longrightarrow R \longrightarrow 0$$

with linear section $s : R \longrightarrow S$, there exists a morphism of algebras $f : R \longrightarrow S$.

An equivalent condition for R to be quasi-free is that it has cohomological dimension ≤ 1 with respect to Hochschild cohomology. Due to the Morita invariance of this cohomology, if R quasi-free then its matrix algebras $M_n(R)$ are also.

Definition 4.3.6. *A **quasi-free extension** of a locally convex algebra A is a quasi-free algebra R which fits into a linearly-split extension of algebras*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & R \arrow[r, "p"] & A \arrow[r] \arrow[l, dashed, bend left, "s"] & 0 \end{tikzcd} .$$

Fix a complex locally convex algebra A. The tensor algebra TA associated to A is a free (hence quasi-free) algebra because any linear map $A \to B$ extends to a morphism of algebras $TA \to B$. The multiplication ideal $IA = \ker(m : TA \to A)$ yields an exact sequence

$$0 \longrightarrow IA \longrightarrow TA \xrightarrow{m} A \longrightarrow 0.$$

In other words, TA a (quasi-)free extension of the algebra A. Recall that the ring $\Omega^\star A = \bigoplus_{n\geq 0} \Omega^n A$ of differential forms on A is endowed with two differentials :

$$d : \begin{array}{ccc} \Omega^n A & \longrightarrow & \Omega^{n+1} A \\ \omega & \longmapsto & d\omega \end{array} \quad \text{and} \quad b : \begin{array}{ccc} \Omega^n A & \longrightarrow & \Omega^{n-1} A \\ \omega da & \longmapsto & (-1)^n [\omega, a] \end{array}$$

Definition 4.3.7. *Define the X**-complex of** A as the $\mathbb{Z}/2\mathbb{Z}$-graded complex :*

$$X(A) := \begin{tikzcd} A \arrow[r, shift left, "d"] & \Omega^1 A / b(\Omega^2 A) \arrow[l, shift left, "b"] \end{tikzcd} .$$

The main result of the Cuntz-Quillen approach to periodic cyclic homology is the following theorem.

Theorem 4.3.8. *[CQ95] Let R be a quasi-free extension of A. The X-complex of the algebra $\widehat{R} := \varprojlim R/I^n$ is naturally and continuously chain-homotopy equivalent to the periodic cyclic complex of A*

$$\mathrm{CQ} : X(\widehat{R}) \xrightarrow{\sim} \widehat{CC}(A).$$

By definition, derived functors must be $\mathbb{Z}$-graded as Hochschild homology for instance. The strength of this theorem is that it endows the periodic cyclic homology with a derived functor behavior because the choice of a quasi-free extension yields a unique X-complex modulo homotopy.

4.3.3 The Chern character of Cuntz-Quillen

Let A be complete locally convex algebra and R be a quasi-free extension of A. Due to lemma [CQ95], lemma 12.1, any idempotent matrix $e \in M_n(A)$ induces an idempotent matrix $\widehat{e} \in M_n(\widehat{R})$ unique up to conjugation. It verifies the identity

$$d\widehat{e} = b((2\widehat{e}-1)d\widehat{e}d\widehat{e}) \in b(\Omega^2 M_n(\widehat{R}))$$

and defines a 0-th cycle of the X-complex of $M_n(\widehat{R})$. Similarly, the lemma [CQ95], lemma 12.3 states that any invertible matrix $u \in GL_m(A)$ extends to an invertible matrix $\widehat{u} \in GL_m(\widehat{R})$. The element $\widehat{u}^{-1}d\widehat{u} \in \Omega^1 GL_m(\widehat{R})$ defines a 1-cycle for the X-complex of $GL_m(\widehat{R})$ because

$$b(\widehat{u}^{-1}d\widehat{u}) = [u^{-1}, u] = 0 \in GL_m(\widehat{R}).$$

This yields to two morphisms

$$\begin{array}{ccc} K_0(A) & \longrightarrow & H_0\left(X(M_n(\widehat{R}))\right) \\ [e_0]-[e_1] & \longmapsto & [\widehat{e}_0]-[\widehat{e}_1] \end{array} \quad \text{and} \quad \begin{array}{ccc} K_1(A) & \longrightarrow & H_1\left(X(GL_m(\widehat{R}))\right) \\ [u] & \longmapsto & \left[\widehat{u}^{-1}d\widehat{u}\right] \end{array}.$$

The algebra $M_n(\widehat{R})$ is isomorphic to $\widehat{M_n(R)}$ because $M_n(I^m) = M_n(I)^m$. Furthermore $M_n(R)$ is a quasi-free extension of $M_n(A)$ because the sequence

$$0 \longrightarrow M_n(I) \longrightarrow M_n(R) \longrightarrow M_n(A) \longrightarrow 0$$

is exact and $M_n(R)$ is quasi-free when R is quasi-free. Same thing occurs for the algebra $GL_m(\widehat{R})$ which is isomorphic to $\widehat{GL_m(R)}$, and for $GL_m(R)$ which defines a quasi-free extension of the algebra $GL_m(A)$. The Cuntz-Quillen theorem 4.3.8 and the Morita invariance of periodic cyclic homology establish the isomorphism

$$\mathrm{tr}\circ\mathrm{CQ} : H_\bullet\left(X(M_n(\widehat{R}))\right) \xrightarrow{\sim} HP_\bullet(M_n(A)) \xrightarrow{\sim} HP_\bullet(A).$$

Since the chain-homotopy equivalence of the Cuntz-Quillen theorem is natural, these constructions don't depend on the choice of the quasi-free extension of A. This yields to the definition of the **Cuntz-Quillen Chern character**, as a natural and canonical morphism.

$$Ch_{\mathrm{CQ}} : K_\bullet(A) \longrightarrow HP_\bullet(A) \tag{25}$$

defined on the even and odd K-groups by the formulas

$$Ch_{\mathrm{CQ}}([e_0]-[e_1]) = [\mathrm{tr}(\mathrm{CQ}(\widehat{e_0}))] - [\mathrm{tr}(\mathrm{CQ}(\widehat{e_1}))] \quad \text{and} \quad Ch_{\mathrm{CQ}}([u]) = [\mathrm{tr}(\mathrm{CQ}(\widehat{u}^{-1} d\widehat{u}))].$$

Theorem 4.3.9. *[CQ95] If $A = \mathscr{C}^\infty(M)$ is the algebra of smooth functions on a compact manifold M, the Chern character of Cuntz-Quillen is compatible with the classical Chern character in the sense that the following diagram commutes*

$$\begin{tikzcd} K_\bullet(\mathscr{C}^\infty(M)) \arrow[r, "\sim"] \arrow[d, "Ch_{\mathrm{CQ}}"'] & K^\bullet(M) \arrow[d, "Ch"] \\ HP_\bullet(\mathscr{C}^\infty(M)) \arrow[r, "\sim"'] & H^\bullet_{dR}(M) \end{tikzcd}$$

5 Bivariant K-theory

Sommaire

Bivariant K-theory of $C^\star$-algebras provides a unifying framework that simultaneously generalizes K-theory and K-homology. Introduced by G. Kasparov [Kas81] [Kas88], KK-theory assigns to a pair of $C^\star$-algebras (A, B) an abelian group $KK(A, B)$, whose elements can be thought as generalized morphisms from A to B. The two variables of this bifunctor do not play exactly symmetric roles. The first one is more algebraic in nature while the second one which is more analytic. In particular K-theory and K-homology appear as the special cases $K_\bullet(A) \simeq KK(\mathbb{C}, A)$ and $K^\bullet(A) \simeq KK(A, \mathbb{C})$. Moreover, the Kasparov product

$$- \times - : KK(A, B) \times KK(B, C) \longrightarrow KK(A, C)$$

endows the theory with a powerful composition mechanism. The existence of this product and some of its properties have been proved by Kasparov. It is associative and natural, and it underlies many deep constructions, including formulations of the Baum-Connes conjecture.

5.1 KK-theory à la Cuntz

Cuntz's interpretation is based on the idea that bivariant K-theory should be the universal theory satisfying certain natural axioms : homotopy invariance, stability, and split exactness. The resulting bivariant theory groups $KK(A, B)$ can then be described in terms of homotopy classes of $\star$-homomorphisms from the Cuntz' algebra qA to $B \otimes_{C^\star} \mathcal{K}$, where $\mathcal{K}$ denote the $C^\star$-algebra of compact operators on a infinite dimensional Hilbert space. In this setting, the Kasparov product corresponds to composition of morphisms, translating for instance questions of invertibility to computation of Kasparov products.

5.1.1 The Cuntz' algebra

Let A and B be two complex algebras. The **algebraic free product** $A \star B$ is the algebra generated by words with letters alternating between A and B, modulo the relations internal to each algebra :

$$A \star B := A \oplus B \oplus (A \otimes B) \oplus (B \otimes A) \oplus (A \otimes B \otimes A) \oplus (B \otimes A \otimes B) \oplus \cdots$$

It is naturally equipped with two canonical inclusions $\iota_0 : A \to A \star B$ and $\iota_1 : B \to A \star B$. This free product should be thought of as the *universal* algebra generated by A and B with the following universal property.

Proposition 5.1.1. *The algebraic free product defines a finite coproduct on the category of complex algebras :*

$$\begin{array}{ccc} \mathrm{Mor}_{\mathbb{C}\text{-}\boldsymbol{Alg}}(A, C) \times \mathrm{Mor}_{\mathbb{C}\text{-}\boldsymbol{Alg}}(B, C) & \xrightarrow{\sim} & \mathrm{Mor}_{\mathbb{C}\text{-}\boldsymbol{Alg}}(A \star B, C) \\ (\phi, \psi) & \longmapsto & \phi \star \psi \end{array}$$

where the morphism $\phi \star \psi$ is defined such that $(\phi \star \psi) \circ \iota_0 = \phi$ and $(\phi \star \psi) \circ \iota_1 = \psi$.

Definition 5.1.2. *We define the $C^\star$**-free product** QA to be the completion of the algebraic free product $A \star A$ such that for any $C^\star$-algebra C, the following is an isomorphism :*

$$\mathrm{Mor}_{C^\star\text{-}\boldsymbol{Alg}}(A, C) \times \mathrm{Mor}_{C^\star\text{-}\boldsymbol{Alg}}(A, C) \xrightarrow{\sim} \mathrm{Mor}_{C^\star\text{-}\boldsymbol{Alg}}(QA, C).$$

The $C^\star$-algebra QA stands as the coproduct in the category of $C^\star$-algebras. An argument of density shows that it is also endowed with the two canonical inclusions $\iota_0, \iota_1 : A \to QA$.

Definition 5.1.3. *The **Cuntz's algebra** qA associated to A is defined to be the kernel of the fold map $id \star id : QA \to A$ arising from the pair of identity morphisms $qA :=$ $\ker(id \star id : QA \to A)$.*

This construction $A \rightsquigarrow qA$ can be thought as an endofunctor of the category of $C^\star$-algebras in the sense that any morphism $\phi : A \to B$ of $C^\star$-algebras induces a morphism $q\phi : qA \to qB$ between the respective Cuntz's algebras. The Cuntz's algebra qA fits into the following doubly-split extension of algebras :

$$\begin{tikzcd} 0 \arrow[r] & qA \arrow[r, hook] & QA \arrow[r, "id\star id"] & A \arrow[l, bend right, "{\iota_0, \iota_1}"'] \arrow[r] & 0 \end{tikzcd}$$

Since $(id \star id) \circ \iota_0 = id = (id \star id) \circ \iota_1$ by definition of the $C^\star$-free product, the image of the function $\iota_0 - \iota_1 : A \to QA$ lies in qA. Indeed qA is exactly the two-side ideal generated by differences $\iota_0(a) - \iota_1(a)$ for $a \in A$. The following fundamental theorem is due to J. Cuntz.

Theorem 5.1.4. *[Cun87] For all separable $C^\star$-algebra A, up to stabilization by square matrices $q(qA)$ and qA are homotopy equivalent. In particular $K_\bullet(qA) \simeq K_\bullet(A)$ and $K_\bullet(q\phi) = K_\bullet(\phi)$ for any morphism $\phi : A \to B$ of separable $C^\star$-algebras.*

5.1.2 The Cuntz' identification

The Cuntz' approach was motivated by the search of an algebraic description of the bivariant K-theory. Indeed, K-theory and K-homology groups should both encode differences of $\star$-homomorphisms, and that the kernel qA of the fold map is the universal $C^\star$-algebra in which such differences become genuine algebraic elements. This identifies qA as the natural representing object for bivariant K-theory.

Theorem 5.1.5. *[Cun87] The **bivariant K-theory** or **KK-theory** of a pair (A,B) of separable $C^\star$-algebras identifies with the homotopy classes of continuous morphisms between qA and $B\otimes_{C^\star}\mathcal{K}$:*

$$KK(A,B) \xrightarrow{\sim} \mathrm{Hom}(qA, B\otimes_{C^\star}\mathcal{K})_{/\sim}$$

Remark Originally, the definition of bivariant K-theory of a pair (A,B) was introduced in the 80's by G. Kasparov [Kas81] as the abelian group of homotopy classes of Fredholm (A,B)-bimodules. J. Cuntz showed few years later that this definition of Kasparov modules coincide with the one we gave above [Cun87] [CG24].

Let us describe what $KK(A,\mathbb{C})$ and $KK(\mathbb{C},A)$ look alike. By definition, $KK(A,\mathbb{C})$ is the group of homotopy classes of homomorphisms $qA\to\mathcal{K}(H)$. The definition of the Cuntz algebra implies that such morphisms can be obtained from a pair of morphisms $A\to\mathcal{L}(H)$ whose differences are compact. Given a Fredholm module (ρ,F) over A such that $F^2=1$ 3.2.1, the bounded operators $\rho, F\rho F : A\to B(H)$ are of differences $\rho(a)-F\rho(a)F=(\rho(a)F-F\rho(a))F$ for all $a\in A$, which is compact by definition of a Fredholm module. Hence, any even Fredholm module in the K-homology group $K^0(A)$ produces a class in $KK(A,\mathbb{C})$:

$$\begin{array}{ccc} K^0(A) & \longrightarrow & KK(A,\mathbb{C}) \\ \big[(\rho,F)\big] & \longmapsto & [(\rho\star F\rho F)_{|qA} : qA\to\mathcal{K}] \end{array}. \tag{26}$$

Also, given two idempotent matrices e_0, e_1 with coefficient in A representing a class in the K-theory group $K_0(A)$ induces a pair of homomorphism $\mathbb{C}\to A\otimes_{C^\star}\mathcal{K}$ sending the generator $1\in\mathbb{C}$ to either e_0 or e_1. This construction associates to any idempotent matrix in $K^0(A)$ a class in $KK(\mathbb{C},A)$:

$$\begin{array}{ccc} K_0(A) & \longrightarrow & KK(\mathbb{C},A) \\ [e_0]-[e_1] & \longmapsto & [(e_0\star e_1)_{|q\mathbb{C}} : q\mathbb{C}\to A\otimes_{C^\star}\mathcal{K}] \end{array}. \tag{27}$$

Theorem 5.1.6. *For any $C^\star$-algebra A, these constructions produce isomorphisms $K^0(A)\simeq KK(A,\mathbb{C})$ and $K_0(A)\simeq KK(\mathbb{C},A)$.*

In other words, KK-theory stands as a generalization of both K-theory and K-homology.

Remark Let locally compact group G acts continuously on the $C^\star$-algebra A. Due to the theorem 5.1.6 above, a class in G-equivariant K-theory $K^0_G(A)$ induces a unique morphism $qA \longrightarrow \mathcal{K}$ modulo homotopy, which is G-equivariant. Idem for the G-equivariant K-homology $K_0^G(A)$ where any class induces a unique G-equivariant morphism $q\mathbb{C} \longrightarrow A \otimes_{C^\star} \mathcal{K}$ modulo homotopy. When the group $G = K$ is compact these constructions are isomorphisms. More generally, the **K-equivariant KK-theory groups** $KK_K(A,B)$ can be modeled via homotopy classes of K-equivariant morphisms :

$$KK_K(A,B) \xrightarrow{\sim} \mathrm{Hom}_K(qA, B \otimes_{C^\star} \mathcal{K})_{/\text{homotopy}},$$

which yields for instance to the description $KK_K(\mathbb{C},\mathbb{C}) \simeq \mathrm{Rep}(K)$ via the Peter-Weyl theorem 2.3.1 and more generally $KK_K(A,\mathbb{C}) \simeq K_0^K(A)$ and $KK_K(\mathbb{C},A) \simeq K_K^0(A)$.

Definition 5.1.7. *Take a Hilbert space $\mathcal{H}$ endowed with a unitary representation $\pi : G \to \mathcal{L}(\mathcal{H})$. The G-**smooth** p-**Schatten ideal** of $\mathcal{H}$ is defined to be :*

$$\ell^{p,G}(\mathcal{H}) := \{T \in \ell^p(\mathcal{H}) \mid g \mapsto \pi(g)T\pi(g)^{-1} \text{ is infinitely differentiable for all } g \in G\}.$$

Lemma 5.1.8. *If K is a compact group of a locally compact group G, the trivial class $1 \in KK_K(\mathbb{C},\mathbb{C})$ can be represented by the composition of algebra homomorphisms*

$$\mathbb{1}_K : \; q(q\mathbb{C}) \xrightarrow{(id\star 0)\circ(id\star 0)} \mathbb{C} \xrightarrow{1\mapsto p_1} M_2(\ell^p(\mathcal{H}^K)) \subseteq M_2(\ell^{p,G}(\mathcal{H}))$$

where p_1 is the 2×2-matrix whose top-left entry is the projection onto a one dimensional subspace of K-invariant vectors.

One of the key point of the Cuntz's approach is that it provides a simple description of the Kasparov product. Given A, B and C three $C^\star$-algebras we can *compose* some continuous maps $f : qA \to B \otimes_{C^\star} \mathcal{K}$ and $g : qB \to C \otimes_{C^\star} \mathcal{K}$ in the following way

$$g \circ qf : qA \xrightarrow[5.1.4]{\sim} q(qA) \longrightarrow q(B \otimes_{C^\star} \mathcal{K}) \longrightarrow C \otimes_{C^\star} \mathcal{K} \otimes_{C^\star} \mathcal{K} \simeq C \otimes_{C^\star} \mathcal{K}. \tag{28}$$

Due to the homotopy invariance of KK-theory, $g \circ qf$ defines a class in $KK(A,C)$ which depends only on the homotopy classes of f and g. In other words, this composition extends up to the Kasparov product of the bivariant K-groups

$$\begin{array}{rccc} -\times_B - : & KK(A,B)\times KK(B,C) & \longrightarrow & KK(A,C) \\ & ([f],[g]) & \longmapsto & [f]\times_B [g] := [g\circ qf] \end{array}. \tag{29}$$

The Kasparov product of two K-equivariant KK-theory classes is again a K-equivariant KK-theory class because their *composition* in the sense of (28) is a K-equivariant morphism. In other words, the Kasparov product descends to the K-equivariant model $KK_K(A,B) \times KK_K(B,C) \longrightarrow KK_K(A,C)$.

5.2 The Dirac and dual Dirac elements

Take G be a Lie group and K a maximal compact subgroup of G. The homogeneous space G/K is a complete Riemannian-manifold. The group G acts by left translations on G/K, preserving the metric. It is well known that for any $x \in G$ the exponential map realizes a diffeomorphism $\exp : T_x(G/K) \longrightarrow G/K$ for any $x \in G/K$. **Suppose G/K is even-dimensional**.

We defined before a Dirac element $\alpha \in K^0_G(\mathscr{C}_\tau(G/K))$ (3.3.3) and a dual Dirac element $\beta \in K_0(\mathscr{C}_\tau(G/K))$ (2.2.9). Since the action of G preserves the metric, one can show that the dual Dirac element defines a class in K-equivariant algebraic K-theory $\beta \in K^K_0(\mathscr{C}_\tau(G/K))$. Due to the description (5.1.2), these elements can be viewed as G-equivariant and K-equivariant morphisms respectively :

$$\alpha : q(\mathscr{C}_\tau(G/K)) \longrightarrow \mathscr{K} \text{ and } \beta : q\mathbb{C} \longrightarrow \mathscr{C}_\tau(G/K) \otimes M_2(\mathbb{C}).$$

The apparition of 2×2-square matrices instead of the compact operators $\mathscr{K}$ arises from the definition of the dual Dirac element using matrices of this size (2.2.9).

Theorem 5.2.1 (Kasparov)**.** *The Kasparov product of the dual Dirac element β with the Dirac element α is trivial in both direction :*

$$\beta \times_{\mathscr{C}_\tau(G/K)} \alpha = 1 \in KK_K(\mathbb{C},\mathbb{C}) \simeq \mathrm{Rep}(K) \text{ and } \alpha \times_{\mathbb{C}} \beta = 1 \in KK_K(\mathscr{C}_\tau(G/K), \mathscr{C}_\tau(G/K)).$$

This theorem will be fundamental in this study because in the Cuntz picture, Kasparov products may be thought as composition of the associated homomorphisms. The fact that their products in both direction is trivial realizes their associated homomorphisms as inverse to each others, this will allow us to adapt the Dirac-dual Dirac method in periodic cyclic homology.

Corollary 5.2.2. *The following compositions are homotopy equivalent to the identity at the level of K-theory :*

$$\alpha \circ q\beta : q(q\mathbb{C}) \longrightarrow M_2(\mathscr{K}) \text{ and } \beta \circ q\alpha : q(\mathscr{C}_\tau(G/K)) \longrightarrow \mathscr{C}_\tau(G/K) \otimes M_2(\mathbb{C}).$$

In particular, the followings are inverses to each other, thus isomorphisms

$$\alpha : K_0(\mathscr{C}_\tau(G/K)) \xrightarrow{\sim} K_0(\mathbb{C}) \text{ and } \beta : K_0(\mathbb{C}) \xrightarrow{\sim} K_0(\mathscr{C}_\tau(G/K)).$$

Démonstration. The compositions $\alpha \circ q\beta$ and $\beta \circ q\alpha$ are homotopy equivalent to the products $\beta \times_{\mathscr{C}_\tau(G/K)} \alpha$ and $\alpha \times_{\mathbb{C}} \beta$ by the Cuntz description of the Kasparov product (29). But the Kasparov's theorem above states that these products are trivial, which make these compositions homotopy equivalent to the identity morphism. The homotopy invariance of the K-theory gives $K(\alpha \circ q\beta) = id$ and $K(\beta \circ q\alpha) = id$. The corollary follows from the theorem 5.1.4 and the computations $K(\alpha) \circ K(\beta) = K(\alpha) \circ K(q\beta) = K(\alpha \circ q\beta) = id$ and $K(\beta) \circ K(\alpha) = K(\beta) \circ K(q\alpha) = K(\beta \circ q\alpha) = id$. $\square$

Corollary 5.2.3. *The following morphisms of algebras are K-equivariantly homotopy equivalent to each other (see 5.1.8)*

$$\alpha \circ q\beta : q(q\mathbb{C}) \longrightarrow M_2(\mathscr{K}) \text{ and } \mathbb{1}_K : q(q\mathbb{C}) \longrightarrow M_2(\ell^{p,G}(\mathscr{H})) \subseteq M_2(\mathscr{K}).$$

Démonstration. Since $\beta \times_{\mathscr{C}_\tau(G/K)} \alpha = 1 \in KK_K(\mathbb{C}, \mathbb{C})$, the composition $\alpha \circ q\beta$ and $\mathbb{1}_K$ represents the same class in $KK_K(\mathbb{C}, \mathbb{C})$. The assertion follows from the Cuntz description 5.1.5 of bivariant K-theory. □

6 The Baum-Connes conjecture

Sommaire

A recurrent theme of this thesis has been the importance of reduced group $C^\star$-algebras and the difficulty to compute their K-theory. For groups of geometric origin, geometry can be used via the assembly map to obtain information about $K_i(C_r^\star(G))$.

The guiding philosophy of the assembly map is that the analytic K-theory of a group should not be approached directly, but rather *assembled* from the topological data of proper G-actions. The natural receptacle for such data is the equivariant K-homology of the universal space $\underline{E}G$, which encodes all proper actions of G. The assembly map thus provides a canonical transformation

$$\mu : K_i^G(\underline{E}G) \longrightarrow K_i(C_r^\star(G)),$$

sending geometric cycles — such as equivariant elliptic operators — to analytic classes in the group $C^\star$-algebra. From this perspective, the assembly map generalizes the index theorem : it transforms geometric information into analytic invariants.

In this way, the assembly map serves as a conceptual and computational tool : it is the mechanism that translates accessible topological information into the deep and often inaccessible analytic invariants of noncommutative geometry. The celebrated Baum–Connes conjecture, formulated in the 1980s, asserts that this translation is in fact exact : the assembly map is an isomorphism for all second countable, locally compact groups.

6.1 The assembly map and the Baum-Connes conjecture

The geometric side of the Baum-Connes assembly map encapsulates the data of proper actions of G. A space $\underline{E}G$ equipped with a proper action of G is said to be **universal** if it is paracompact with a metrizable orbit space $\underline{E}G/G$ and if for every proper metrizable G-space X with X/G paracompact, there exists a unique G-equivariant continuous map $X \to \underline{E}G$ up to equivariant homotopy. Such a universal space is unique up to equivariant homotopy. The orbit space $\underline{E}G/G$ encodes the *geometry of proper actions* of the group.

For instance, when G is discrete and torsion free $\underline{E}G$ is nothing else than the universal covering space EG and EG/G is the classifying space BG. In that case the space $\underline{E}G$ captures geometrically the information coming from the group since $K_i^G(\underline{E}G) \simeq K_i(BG)$, and the singular homology of the classifying space BG computes the integer homology of the group G. Also, if K is a compact group, $\underline{E}K$ is homeomorphic to a point $\{pt\}$, which is $K_0^K(\underline{E}K) \simeq KK_K(\mathbb{C},\mathbb{C})$ due to the identification (**??**). Finally, when G is a connected reductive Lie group and K a maximal compact subgroup, the homogeneous space G/K gives a local model for the universal proper space $\underline{E}G$ and we have $K_i^G(\underline{E}G) \simeq K_G^i(\mathscr{C}_0(G/K))$.

The *assembly map* is a function that sends a class of $K_i^G(\underline{E}G)$ to a class of $K_i(C_r^\star(G))$, *assembling* a bridge from *geometric* to *analytic* sides [HR01] and [Val02]. Let us describe the even case more precisely. Take (ρ, F, U) an even G-equivariant Fredholm module on the universal space $\underline{E}G$. The image of the representation ρ on the underlying Hilbert space H, which is $V := \rho(\mathscr{C}_c(\underline{E}G))H \subseteq H$, carries a natural $\mathscr{C}_c(G)$-valued inner product

$$\langle \xi_1, \xi_2 \rangle(g) = \langle \xi_1, U_g \xi_2 \rangle_H.$$

Then, completing V for this product yields a Hilbert $C_r^\star(G)$-module $\mathscr{V}$. The operator F extends as an adjointable operator $\mathscr{F}$ over $\mathscr{V}$. The extended operator $\mathscr{F}$ is invertible modulo compact $C_r^\star(G)$-linear operators, and then *Fredholm* on $\mathscr{V}$. If its kernel and cokernel are finitely generated projective modules over $C_r^\star(G)$, then the **analytic index** of $\mathscr{F}$ equals :

$$\mathrm{Ind}(\mathscr{F}) := [\ker(\mathscr{F})] - [\mathrm{coker}(\mathscr{F})] \in K_0(C_r^\star(G)).$$

It is closely related to the boundary maps appearing at theorem 4.3.3. This construction, known as *Mishchenko-Fomenko index*, depends only on the class of (ρ, F, U) in the equivariant K-homology $K_i^G(\underline{E}G)$ and then defines a map called **assembly map** [MF80] :

$$\begin{array}{rccc} \mu^G : & K_i^G(\underline{E}G) & \longrightarrow & K_i(C_r^\star(G)) \\ & [(\rho, F, U)] & \longmapsto & \mathrm{Ind}(\mathscr{F}) \end{array} \tag{30}$$

Conjecture 1 (Baum-Connes)**.** *The assembly map is an isomorphism.*

As the left hand side tends to be more easily accessible than the right hand side, one usually views the Baum-Connes conjecture as an *"explanation"* of the right hand side. The original formulation of this striking conjecture by A. Connes and P. Baum was done in 1982. It sets up a correspondence between different areas of mathematics as the left-hand side is of geometric nature while the right-hand side is a purely analytical object. Let's draw cases where the assembly map is well understood.

This conjecture doesn't hold yet but has been proved for the several classes of groups by different methods :

- connected reductive Lie groups : uses Arthur's classification of the unitary dual and deep structural properties of the reduced $C^\star$-algebra of the group (by A. Wassermann in 1987) ;
- groups with Haagerup property : relies on *E-theory* and KK-theory analytic tools for the surjectivity of the assembly map while the injectivity uses constructions of *Dirac-dual Dirac elements* [HK97] ;
- groups with Rapid Decay property (RD) and a proper, cocompact, isometric action on a strongly bolic metric space : uses *Banach KK-theory* as an extension of Kasparov's KK-theory, geometric estimates and analytic properties of group actions on strongly bolic metric spaces [Laf98] ;
- groups that admit a finite presentation with only one relation : applies geometric group theory techniques, the RD property and employs some new analytic tools adapted to one-relator groups [BBV99] ;
- algebraic groups on characteristic zero local fields : employs tools from p-adic representation theory and equivariant KK-theory, reducing to well-understood cases [CEN03] ;
- Gromov hyperbolic groups and their subgroups : relies on coarse geometric insights and controlled operator algebra techniques and employs cyclic homology and Chern character methods to relate analytic and geometric K-homology [Pus12] ;
- other cases due to G. Skandalis, B. Bekka, P. de la Harpe et A. Valette, using harmonic analysis, operator-algebraic methods, and representation theory via orbital integrals and character formulas.

The main example for which the conjecture is still not proved is the discrete group $SL_3(\mathbb{Z})$. The conjecture can also be stated *with coefficients* in a $C^\star$-algebra A equipped with an action of G, using crossed product algebras instead of reduced group $C^\star$-algebras.

Conjecture 2 (Baum-Connes with coefficients)**.** *For every locally compact group acting continuously on a $C^\star$-algebra A, the following assembly map is an isomorphism :*

$$\mu_A^G : KK_G^i(\mathscr{C}_0(\underline{E}G), A) \longrightarrow K_i(A \rtimes_{\alpha,r} G).$$

Some counter-examples have been found for this extended conjecture by N. Higson, V. Lafforgue and G. Skandalis on graph groups [HLS02], using the works of M-L. Gromov.

Theorem 6.1.1 (Green-Julg)**.** *The Baum-Connes conjecture is true for compact groups. In other words, if K is a compact group acting continuously on a $C^\star$-algebra A, the assembly map is an isomorphism :*

$$\mu_A^K : KK_K^i(\mathbb{C}, A) \xrightarrow{\sim} K_i(A \rtimes_\alpha K).$$

6.2 The Connes-Kasparov theorem

Let G be a connected real reductive Lie group and K a maximal compact subgroup. The Connes-Kasparov theorem stands a *Lie-version* of the Baum-Connes assembly map. We fix G to be a connected reductive Lie group and K as a maximal compact subgroup of G. For any $C^\star$-algebra A, their respective assembly maps yield to the following construction :

$$\begin{tikzcd} K_i(A \rtimes_\alpha K) \arrow[r, dashed] & K_{i+\dim(G/K)}(A \rtimes_{\alpha,r} G) \\ KK_K^i(\mathbb{C}, A) \arrow[u, "\mu_A^K", "\sim"'] \arrow[r, "\text{Thom}", "\sim"'] & KK_G^{i+\dim(G/K)}(\mathscr{C}_0(G/K), A) \arrow[u, "\mu_A^G"'] \end{tikzcd}$$

where the bottom arrow can be thought as a Thom isomorphism and the vertical left arrow is an isomorphism due to the Green-Julg theorem above. The assembly map of G then bridges the K-theoretical invariants of the reduced crossed product algebras $A \rtimes_\alpha K$ and $A \rtimes_{\alpha,r} G$. The dashed arrow is called Dirac-induction :

$$\text{D-Ind}_A : K_i(A \rtimes_\alpha K) \longrightarrow K_{i+\dim(G/K)}(A \rtimes_{\alpha,r} G). \tag{31}$$

The Baum-Connes conjecture for reductive Lie groups is equivalent to the fact that the Dirac-induction is an isomorphism.

Theorem 6.2.1 (Connes-Kasparov)**.** *When $A = \mathbb{C}$, the Dirac-induction is an isomorphism :*

$$\text{D-Ind}_{\mathbb{C}} : K_i(C^\star(K)) \xrightarrow{\sim} K_{i+\dim(G/K)}(C_r^\star(G)).$$

For reductive Lie groups, the Connes-Kasparov isomorphism was proved in two ways by A. Wassermann in 1987 and V. Lafforgue in 1998 [Laf98]. A third way to prove the Connes-Kasparov conjecture has long been suspected to exist. As A. Connes and N. Higson insisted, this meant that the Connes-Kasparov isomorphism could be the non-commutative geometric counterpart of a representation theoretic phenomenon. The reformulation of the Connes-Kasparov conjecture in terms of deformations reflects the observations by G. W. Mackey and leads naturally to the *Mackey's Analogy in K-theory*.

6.3 Mackey analogy

The Mackey analogy refers to a correspondence between the tempered representation theory of a real reductive group G and that, much simpler, of its associated Cartan motion group G_0. It takes the form of a bijection, due to Higson in the complex case and Afgoustidis in the general case [Afg15] [Afg20] [Afg19], between the tempered duals of these groups, which preserves certain invariants. This analogy, reformulated in operator-algebraic terms, provides insight into the structure of the unitary dual and clarifies the role of crossed product algebras in representation theory. In this part we review the Cartan motion group, explain its relation to equivariant K-theory, and discuss Connes-Higson's deformation picture of the assembly map, which encapsulates the geometric content of the Connes-Kasparov theorem. This section is inspired by [Hig08].

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of G. The **Cartan motion group** is defined as the semidirect product

$$G_0 = K \ltimes \mathfrak{p},$$

where K acts on $\mathfrak{p}$ via the adjoint representation. Although G and G_0 are diffeomorphic as manifolds, they carry fundamentally different group structures.

Mackey observed that the representation theory of G should be closely related to that of G_0. This idea was later made precise by Higson and Afgoustidis through a bijection between the tempered dual of G and the unitary dual of G_0. From the operator-algebraic viewpoint, this analogy suggests comparing the K-theory of $C^\star_r(G)$ with that of $C^\star(G_0)$. Since G_0 is an extension of K by a vector abelian group, it is amenable and satisfies $C^\star_r(G_0) = C^\star(G_0)$. Bott periodicity then yields a natural identification

$$K_0(C^\star(G_0)) \simeq R(K).$$

Under this identification, the Connes-Kasparov theorem may be interpreted as the comparison map induced by a deformation from G_0 to G, a perspective formalized by the Connes-Higson deformation picture of the assembly map. We define the **deformation bundle** as the disjoint union :

$$\nu(K,G) := \nu K \times \{0\} \sqcup G \times \mathbb{R}^\times$$

were νK, which is homeomorphic to G_0, denotes the quotient of the restriction of the tangent bundle over G by the tangent bundle over K. The **deformation group** G_t is defined to be the fiber of the deformation bundle over $t \in \mathbb{R}$. It is natural to think of $\{G_t\}_{t\in\mathbb{R}}$ as a *continuous* family of groups which interpolates between $G_1 = G$ and G_0.

Lemma 6.3.1. *Let f be a compactly supported function over $\nu(K,G)$ and denote by f_t its compactly supported restriction on the fiber G_t thought as a function over $C^\star_r(G_t)$. The map $t \mapsto \|f_t\|$ is continuous on $\mathbb{R}$.*

This lemma makes the field $\{C^\star_r(G_t)\}_{t\in\mathbb{R}}$ a continuous field of $C^\star$-algebras. Denote by $\mathscr{C}$ the $C^\star$-algebra of continuous sections of the restriction of this continuous field $\{C^\star_r(G_t)\}_{t\in\mathbb{R}}$ to the interval $[0,1]$. Thus $\mathscr{C}$ is the completion of the fiberwise convolution algebra of smooth, compactly supported functions on $\nu(K,G)_{|[0,1]}$ in the norm

$$\|f\|_{\mathscr{C}} := \sup\{\|f_t\| \mid t \in [0,1]\}.$$

The $C^\star$-algebra $\mathscr{C}$ encapsulates the whole deformation from the reduced $C^\star$-algebra of G_0 to the reduced $C^\star$-algebra of $G = G_1$. It is naturally equipped with two evaluation maps at $t = 0, 1$:

$$C^\star(G_0) \xleftarrow{\alpha_0} \mathscr{C} \xrightarrow{\alpha_1} C_r^\star(G).$$

When both of these maps are quasi-isomorphisms in K-theory the Connes-Kasparov conjecture holds. As the topology of $\mathscr{C}$ suggests it, it is easier to show that α_0 is a quasi-isomorphism in K-theory, and it is even feasible to find an inverse of it. The tougher proof for the other map α_1 have been proposed by A. Connes and N. Higson in 1990 and recently reformulated by A. Afgoustidis in 2019.

Theorem 6.3.2 (Connes-Higson)**.** *The map $\alpha_1 \circ \alpha_0^{-1}$ defines a quasi-isomorphism :*

$$K_i(C^\star(G_0)) \xrightarrow{\sim} K_i(C_r^\star(G)).$$

With coefficients in an $C^\star$-algebra A endowed with an action of G, the evaluations at $t = 0$ and $t = 1$ yield a morphism

$$K_i(A \rtimes_\alpha K) \underset{\sim}{\xrightarrow{\text{Thom}}} K_{i+\dim(G/K)}(A \rtimes_\alpha G_0) \xrightarrow{\alpha_1 \circ \alpha_0^{-1}} K_{i+\dim(G/K)}(A \rtimes_{\alpha,r} G)$$

which is the representation theoretic counterpart of the Dirac-induction (31). The proof that this morphism is an isomorphism is equivalent to the Baum-Connes conjecture with coefficients for reductive Lie groups. The Mackey analogy in K-theory proposes another approach to solve this conjecture.

7 Morita equivalences of Banach crossed product algebras

Sommaire

Consider as before a real reductive group G with maximal compact subgroup K, and a complete locally convex G-algebra A. In this section we establish the isomorphism

$$HP_\bullet(\mathscr{C}_c^\infty(G/K,A)\rtimes G)\xrightarrow{\sim} HP_\bullet(A\rtimes K). \tag{32}$$

To do so, we will use the notion of *quasi-Morita equivalence* developed by J. Cuntz and D. Quillen [Cun98], [CQ95] and [CQ95] to obtain periodic cyclic complexes that are homotopy equivalent. This idea relies on the derived functor analogy of the periodic cyclic homology via quasi-free extensions stated in theorem 4.3.8. The datum of two G-algebras that are quasi-Morita equivalent yield to quasi-free extensions of the same algebra, which will canonically and naturally identifies the periodic cyclic complexes of their G-crossed product algebras. In other words, the transformation $\widehat{CC}(-\rtimes G)$ viewed as a functor from the category of locally convex G-algebras to the chain-homotopy category is stable under equivariant quasi-Morita equivalences. In the statement 7.3.3, we show that $(\mathscr{C}_c^\infty(G/K,A),\mathscr{C}_c^\infty(G,A)\rtimes K)$ and $(A,\mathscr{C}_c^\infty(G,A)\rtimes G)$ are pairs of G-equivariantly quasi-Morita equivalent algebras, which stands as a weak version of the Green-Imprimitivity theorem 7.3.3 [Wil07] for Fréchet algebras. The periodic cyclic complexes of their G-crossed product and K-crossed product respectively are homotopy equivalent, which leads to the isomorphism (32) above.

At the end of the section, the theorem 7.3.5 will realize the smooth Clifford module $\mathscr{C}_\tau^\infty(G/K)$ as quasi-Morita equivalent to the algebra of compactly-supported smooth functions $\mathscr{C}_c^\infty(G/K)$.

7.1 The Cuntz-Quillen method

Fix a complex algebra A. Let V and W be two complex vector spaces which fit to a split exact sequence of vector spaces

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "i"] & W \arrow[r, "p"] & A \arrow[l, dashed, bend left, "s"] \arrow[r] & 0. \end{tikzcd} \tag{33}$$

We will denote by TV, TW and TA for the tensor algebras associated to these vector spaces. We write ι for the canonical inclusion $\iota : V \to TV$ and $IA \subseteq TA$ for the kernel of the multiplication $m : TA \to A$. Consider $I = \ker(m \circ (0 \star id) : TV \star TA \to A)$ a two-side ideal of the free product $TV \star TA$, and $J = \ker(m \circ Tp : TW \to A)$ an ideal of TW.

Lemma 7.1.1. *The free homomorphism $Ti \star Ts : TV \star TA \longrightarrow TW$ is an isomorphism of filtrated algebras for the I-adic filtration on $TV \star TA$ and the J-adic filtration on TW.*

Démonstration. To show this free homomorphism is an isomorphism, we construct an inverse. The linear map $\phi : w \in W \longmapsto (w - (s \circ p)(w)) + p(w) \in TV \star TA$ extends to a morphism of algebras $id \star Tp : TW \longrightarrow TV \star TA$ by universal property of tensor algebras. Since $Tp \circ Ts = id$, it is clear that $(id \star Tp) \circ (Ti \star Ts) = id_{TV \star TA}$. On the other side

$$((Ti \star Ts) \circ \phi)(w) = (Ti \star Ts)((w - (s \circ p)(w)) + p(w)) = i(w - (s \circ p)(w)) + (s \circ p)(w) = w,$$

which implies that $(Ti \star Ts) \circ (id \star Tp) = id_{TW}$. To check that $Ti \star Ts$ preserves the respective filtrations it suffices to remark that the following diagram commutes :

$$\begin{tikzcd} TV \star TA \arrow[r, "0 \star id"] \arrow[d, "id \star Ts"', "\sim"] & TA \arrow[r, "m"] \arrow[d, equal] & A \arrow[d, equal] \\ TW \arrow[r, "Tp"'] & TA \arrow[r, "m"'] & A \end{tikzcd}$$

and that I is the kernel of the top arrow while J is the kernel of the bottom one. It ends up the proof. $\square$

Corollary 7.1.2. *The morphisms $Tp : TW \to TA$ and $Ts : TA \to TW$ preserve the J-adic and IA-adic filtrations and are inverse to each other modulo filtrated preserving smooth homotopy.*

Démonstration. First of all, let us check that the morphisms Ts and Tp preserve the filtrations. If $j \in J \subseteq TW$ we get $m(Tp(j)) = (m \circ Tp)(j) = 0$, *i.e.* $Tp(j) \in IA$. On the other side, if $a \in IA \subseteq TA$, $(m \circ Tp)(Ts(a)) = (m \circ T(p \circ s))(a) = m(a) = 0$ which is that $Ts(a) \in J$. In particular we showed that Ts and Tp are continuous for the adic-topologies.
Now we will study their composition. By definition of s and p, it is clear that $Tp \circ Ts =$

id_{TA}. Also, $(Ts\circ Tp)(Ti\star Ts) = T(s\circ p\circ i)\star T(s\circ p\circ s) = 0\star Ts = (Ti\star Ts)(0\star id)$ and the morphism $Ts\circ Tp$ fits into the following commutative diagram :

$$\begin{tikzcd} TV\star TA \arrow[r, "Ti\star Ts", "\sim"'] \arrow[d, "0\star id"'] & TW \arrow[d, "Ts\circ Tp"] \\ TV\star TA \arrow[r, "Ti\star Ts", "\sim"'] & TW \end{tikzcd}$$

Consider the family of linear maps $h_t : V \to V, v \mapsto tv$ for $t \in [0,1]$ and thus the morphisms $T(h_t)\star id : TV\star TA \to TV\star TA$. It is clear that the homotopy preserves the filtrations and that it is smooth with respect to the parameter $t\in[0,1]$. Since it realizes an homotopy between $0\star id$ and the identity, of $TV\star TA$, it realizes through $Ti\star Ts$ an homotopy between $Ts\circ Tp$ and the identity of TW. The assertion follows. □

Theorem 7.1.3 (Cuntz-Quillen)**.** *Let the assumptions of (33) be understood. Let $\widehat{TW}$ be the J-adic completion of TW and $\widehat{TA}$ be the IA-adic completion of TA. Then there exists canonical chain homotopy equivalences between the respective X-complexes and the periodic cyclic complex of A*

$$X(\widehat{TW}) \xrightarrow{X(Tp)} X(\widehat{TA}) \xrightarrow{\sim} \widehat{CC}(A).$$

Démonstration. By [CQ95][Corollary 9.4.b)], the homotopy equivalence $Tp : TW \xrightarrow{\sim} TA$ induces an homotopy equivalence between the associated towers of complexes because TA and TW are quasi-free algebras. Hence we get an chain-homotopy equivalence $X(\widehat{TW}) \xrightarrow{\sim} X(\widehat{TA})$. The left arrow is a consequence of theorem 4.3.8 because TA is a (quasi-)free extension of A. □

Remark Recall the description of the periodic cyclic complexes as a derived functor for quasi-free extensions 4.3.8. The proof above is not as easy as the verification that TW is a quasi-free extension of A because TW is a topological algebra and it is necessary to verify that the underlying homotopies respect its toplogy.

7.2 Quasi-Morita equivalence

Definition 7.2.1. *[Cun98] Let C be a complete, locally convex algebra. We say that two sub-algebras $A, B \subseteq C$ are **quasi-Morita equivalent** if there exist E and F two closed linear subspaces of C such that the multiplication $m : C\otimes_\pi C \to C$ descends to surjections*

$$E\otimes_\pi F \twoheadrightarrow A \quad \textit{and} \quad F\otimes_\pi E \twoheadrightarrow B$$

with bounded linear sections $s_A : A \to E\otimes_\pi F$ and $s_B : B \to F\otimes_\pi E$.
*They are G-**equivariantly quasi-Morita equivalent** to each other when a Lie group G acts smoothly on the algebra C preserving the sub-algebras and sub-spaces A, B and E, F, respectively.*

Lemma 7.2.2. *Two complete, locally convex algebras A and B are quasi-Morita equivalent if it exists a (A,B)-bimodule E and a (B,A)-bimodule F with linearly-split onto morphisms of A-bimodules and B-bimodules respectively*

$$E \otimes_\pi F \twoheadrightarrow A \ \ \textit{and} \ \ F \otimes_\pi E \twoheadrightarrow B.$$

Démonstration. Consider the the matrix vector space $C = \begin{pmatrix} A & E \\ F & B \end{pmatrix}$. The assumptions of the statement make C be a matrix-type algebra containing A and B as sub-algebras. The multiplication on C descends to the bimodule epimorphisms $E \otimes_\pi F \twoheadrightarrow A$ and $F \otimes_\pi E \twoheadrightarrow B$ and we have linear sections of these by hypothesis. The assertion follows. □

Theorem 7.2.3 (Cuntz). *[CQ95] If A and B are two quasi-Morita equivalent complete locally convex algebras, then there exists a bounded chain homotopy equivalence*

$$\widehat{CC}(A) \xrightarrow{\sim} \widehat{CC}(B).$$

Démonstration. The theorem 7.1.3 applied to $W = E \otimes_\pi F$ and $W = F \otimes_\pi E$ respectively, we obtain two chain-homotopy equivalences $X(\widehat{T(E \otimes_\pi F)}) \xrightarrow{\sim} \widehat{CC}(A)$ and $X(\widehat{T(F \otimes_\pi E)}) \xrightarrow{\sim} \widehat{CC}(B)$. Consider the following linear map

$$\begin{array}{ccc} X(T(E \otimes_\pi F)) & \xrightarrow{\Theta} & X(T(F \otimes_\pi E)) \\ (e_0 \otimes f_0) \otimes (e_1 \otimes f_1) \otimes \cdots \otimes (e_n \otimes f_n) & \longmapsto & (f_n \otimes e_0) \otimes (f_0 \otimes e_1) \otimes \cdots \otimes (f_{n-1} \otimes e_n) \\ (e_0 \otimes f_0) \otimes \cdots \otimes (e_n \otimes f_n) d(e_{n+1} \otimes f_{n+1}) & \longmapsto & (f_{n+1} \otimes e_0) \otimes \cdots \otimes (f_{n+1} \otimes e_n) d(f_n \otimes e_{n+1}) \end{array}.$$

According to [CQ95] and [Cun98], it defines a continuous isomorphism for the adic and locally convex topologies. Since these X-complexes are dense in their respective adic completions, the assertion follows. □

The notion of G-equivariant quasi-Morita enables us to compare not only their respective periodic cyclic complexes but also the ones of their crossed product algebras. Initially proved for the action of a discrete group by J. Cuntz, the proof extends to the following framework.

Theorem 7.2.4. *If A and B are two G-equivariantly quasi-Morita equivalent complete locally convex algebras, then there exists a bounded chain homotopy equivalence*

$$\widehat{CC}(A \rtimes G) \xrightarrow{\sim} \widehat{CC}(B \rtimes G).$$

Démonstration. We take $E, F \subseteq C$ the G-invariant linear subspaces realizing the A and B as G-equivariantly quasi-Morita equivalent algebras. Write δ_e the atomic probability measure centered at the origin of the group G. It fits into the extension

$$0 \longrightarrow \mathscr{C}_c^\infty(G) \longrightarrow M(G) \longrightarrow \mathbb{R} \cdot \delta_e \longrightarrow 0.$$

After tensoring with C we get $0 \longrightarrow C \rtimes G \longrightarrow C' \longrightarrow C \cdot \delta_e \longrightarrow 0$. Since A and B are supposed G-stable the algebras $A \rtimes G$ and $B \rtimes G$ are sub-algebras of $C \rtimes G \subseteq C'$.

Take $E' = \mathscr{C}_c^\infty(G,E)$ and $F' = F\cdot\delta_e$ two subspaces of C'. Then one can check that the multiplication on C' descend to an epimorphism $E'\otimes_\pi F' \twoheadrightarrow A\rtimes G$ and $F'\otimes_\pi E' \twoheadrightarrow B\rtimes G$ with sections

$$\begin{array}{rccc} s'_{A\rtimes G}: & A\rtimes G & \longrightarrow & E'\otimes_\pi F' \\ & f & \longmapsto & \left[g\mapsto (id\otimes\pi(g^{-1}))(s_A(f(g)))\cdot\delta_e\right] \end{array}$$

$$\begin{array}{rccc} s'_{B\rtimes G}: & B\rtimes G & \longrightarrow & F'\otimes_\pi E' \\ & f & \longmapsto & (\delta_e\otimes id)(s_B(f)) \end{array}$$

Which means that $A\rtimes G, B\rtimes G \subseteq \mathscr{C}$ are quasi-Morita equivalent. The assertion follows from the previous theorem. $\square$

7.3 Application to crossed product algebras

7.3.1 A weak Green's Imprimitivity theorem in a Fréchet framework

Let G be a real reductive group and K a maximal compact subgroup of G. Choose on G a left-invariant Haar measure. Let A be a unital Fréchet algebra endowed with a smooth action γ of G. The Lie group $G\times K$ acts on the pointwise multiplication algebra $\mathscr{C}_c^\infty(G,A)$ via two different actions α and β defined for $f\in\mathscr{C}_c^\infty(G,A)$, $(g,k)\in G\times K$ and $s\in G$ by :

$$(\alpha_{(g,k)}\cdot f)(s) := \gamma_g(f(g^{-1}sk)) \;\text{ and }\; (\beta_{(g,k)}\cdot f)(s) = \gamma_h(f(k^{-1}sg)).$$

Lemma 7.3.1. *The crossed products $\mathscr{C}_c^\infty(G,A)\rtimes_\alpha(G\times K)$ and $\mathscr{C}_c^\infty(G,A)\rtimes_\beta(G\times K)$ are isomorphic.*

Démonstration. The linear map $\Phi:\mathscr{C}_c^\infty(G,A)\longrightarrow\mathscr{C}_c^\infty(G,A)$ defined by the formula $\Phi(f)(s)=\gamma_s(f(s^{-1}))$ is an isomorphism of vector spaces which commutes with the pointwise multiplication on $\mathscr{C}_c^\infty(G,A)$:

$$\Phi(f\cdot g)(s) = \gamma_s(f(s^{-1})\cdot g(s^{-1})) = \gamma_s(f(s^{-1}))\cdot\gamma_s(g(s^{-1})) = \Phi(f)(s)\cdot\Phi(g)(s).$$

Moreover, it transports the $G\times K$-action α to the $G\times K$-action β :

$$\Phi(\alpha_{(g,k)}\cdot f)(s) = \gamma_s((\alpha_{(g,k)}\cdot f)(s^{-1})) = \gamma_{sg}(f(g^{-1}s^{-1}k)) = \gamma_h(\Phi(f)(k^{-1}sg)) = (\beta_{(g,k)}\cdot\Phi(f))(s).$$

The assertion follows. $\square$

Remark The crossed product algebras by a product of groups can be decomposed as a composition of crossed products. Writing λ for the left-regular representation $(\lambda_g f)(s) = f(g^{-1}s)$ and ρ for the right-regular represention $(\rho_g f)(s) = f(sg)$ of G on

$\mathscr{C}_c^\infty(G)$, the previous lemma yields a commutative diagram of isomorphisms :

$$\begin{tikzcd} \mathscr{C}_c^\infty(G,A)\rtimes_\alpha(G\times K) \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathscr{C}_c^\infty(G,A)\rtimes_\beta(G\times K) \arrow[d, "\sim"] \\ (\mathscr{C}_c^\infty(G,A)\rtimes_{\rho,id}K)\rtimes_{\lambda,\gamma}G \arrow[r, "\sim"'] & (\mathscr{C}_c^\infty(G,A)\rtimes_{\rho,id}G)\rtimes_{\lambda,\gamma}K \end{tikzcd} . \tag{34}$$

In particular, if we show that for the G-action $\lambda\times\gamma$ the algebras $\mathscr{C}_c^\infty(G,A)\rtimes_{\rho,id}K$ and $\mathscr{C}_c^\infty(G/K,A)$ are G-equivariantly quasi-Morita equivalent and that $\mathscr{C}_c^\infty(G,A)\rtimes_{\rho,id}G$ is K-equivariantly quasi-Morita equivalent to A, we would get the expected homotopy equivalence (32) in periodic cyclic homology.

Fix a closed subgroup $H<G$. Consider the manifold $Y(H)=\{(x,y)\in G\times G \mid xH=yH\}\subseteq G\times G$ and the space $\mathscr{C}_c^\infty(Y(H),A)$ of smooth A-valued functions with compact support on $Y(H)$. It is equipped with a product and a smooth G-action defined by :

$$(f_1\cdot f_2)(x,y)=\int_H f_1(x,xs)f_2(xs,y)ds \text{ and } (s\cdot f)(x,y)=\gamma_s(f(s^{-1}x,s^{-1}y)).$$

It corresponds the algebra of smooth A-valued functions with compact support over the bundle-groupoid associated to $G\to G/H$.

Lemma 7.3.2. *For every closed subgroup $H<G$, the following linear map is an isomorphism of G-algebras*

$$\begin{array}{cccc} \Psi: & \mathscr{C}_c^\infty(Y(H),A) & \longrightarrow & \mathscr{C}_c^\infty(G,A)\rtimes_{\rho,id}H \\ & f & \longmapsto & [(g,h)\mapsto f(g,gh)] \end{array}$$

where the G-action on the $\mathscr{C}_c^\infty(G,A)\rtimes_{\rho,id}H$ is given by $\lambda\times\gamma$.

Démonstration. It is clear that the linear map Ψ realizes an isomorphism of vector spaces. It commutes with the respective products because for $f_1,f_2\in\mathscr{C}_c^\infty(Y(H),A)$ and $(g,h)\in G\times H$:

$$\Psi(f_1\cdot f_2)(g,h)=(f_1\cdot f_2)(g,gh)=\int_G f_1(g,s)f_2(s,gh)d=\int_G f_1(g,gs)f_2(gs,gh)s$$

$$(\Psi(f_1)\star\Psi(f_2))(g,h)=\int_G \Psi(f_1)(g,s)\Psi(f_2)(gs,s^{-1}h)ds=\int_G f_1(g,gs)f_2(gs,gh).$$

The morphism Ψ is also G-equivariant because for $s\in G$, $f\in\mathscr{C}_c^\infty(Y(H),A)$ and $(g,h)\in G\times H$:

$$\Psi(s\cdot f)(g,h)=(s\cdot f)(g,gh)=\gamma_s(f(s^{-1}g,s^{-1}gh))=\gamma_s(\Psi(f)(s^{-1}g,h))=(s\cdot\Psi(f))(g,h).$$

The assertion follows. $\square$

Theorem 7.3.3. *The following pairs are G-equivariantly quasi-Morita equivalent algebras :*

1. $\mathscr{C}_c^\infty(G/K, A)$ *and* $\mathscr{C}_c^\infty(Y(K), A)$ *;*
2. A *and* $\mathscr{C}_c^\infty(Y(G), A)$.

Démonstration. We will show that $\mathscr{A} := \mathscr{C}_c^\infty(G/H, A)$ and $\mathscr{B} := \mathscr{C}_c^\infty(Y(H), A)$ are quasi-Morita equivalent when $H = K$ and $H = G$, using the lemma 7.2.2. Consider $E = F = \mathscr{C}_c^\infty(G, A)$. They are respectively equipped with a $(\mathscr{A}, \mathscr{B})$-bimodule and $(\mathscr{B}, \mathscr{A})$-bimodule structures given, for $(a, b, e, f) \in \mathscr{A} \times \mathscr{B} \times E \times F$, by :

$$(a \cdot e \cdot b)(g) := \int_H a(gH)e(gh)b(gh, g)dh \text{ and } (b \cdot f \cdot a)(g) := \int_H b(g, gh)f(gh)a(gH)dh.$$

The structures of right $\mathscr{A}$-module on E and left $\mathscr{A}$-module on F obviously commute with the pointwise product on $\mathscr{A}$. We verify the structures of left $\mathscr{B}$-module on E and right $\mathscr{B}$-module on F commute with the product on $\mathscr{B}$:

$$\begin{aligned}
(e \cdot (b \cdot b'))(g) &= \int_H e(gh)(b \cdot b')(gh, g)dh \\
&= \int_H \int_H e(gh)b(gh, gs)b'(gs, g)dsdh \\
&= \int_H (e \cdot b)(gs)b'(gs, g)ds \\
&= ((e \cdot b) \cdot b')(g)
\end{aligned}$$

$$\begin{aligned}
((b \cdot b') \cdot f)(g) &= \int_H (b \cdot b')(g, gh)f(gh)dh \\
&= \int_H \int_H b(g, gs)b'(gs, gh)f(gh)dsdh \\
&= \int_H b(g, gs)(b' \cdot f)(gs)ds \\
&= (b \cdot (b' \cdot f))(g)
\end{aligned}$$

Define also two epimorphisms of A-bimodules an B-bimodules respectively

$$\begin{array}{rccc}
p_{\mathscr{A}}: & E \otimes_\pi F & \longrightarrow & \mathscr{A} \\
& (e, f) & \longmapsto & \left[gH \mapsto \int_H e(gh)f(gh)dh\right]
\end{array}$$

$$\begin{array}{rccc}
p_{\mathscr{B}}: & F \otimes_\pi E & \longrightarrow & \mathscr{B} \\
& (f, e) & \longmapsto & [(x, y) \mapsto f(x)e(y)]
\end{array}.$$

It remains to find linear-sections of these morphisms.
1) Case $H = K$. Consider a filtration of G/K by balls $B(K, r_n)$ centered on $K \in G/K$ of strictly increasing radius $r_n > 0$ with $\lim_{n \to \infty} r_n = \infty$. Fix χ_n to be a smooth function with compact support in $B(K, r_{n+1})$ and such that $\chi_n \equiv 1$ on $B(K, r_n)$. By convention $\chi_{-1} \equiv 0$ on G/K. Then any $a \in \mathscr{C}_c^\infty(G/K)$ can be written as a finite sum :

$$a = \sum_{n \geq 0} a(\chi_n - \chi_{n-1}).$$

Define the linear map :

$$\begin{array}{rccc} s_{\mathscr{A}}: & \mathscr{A} & \longrightarrow & E\otimes_\pi F = \mathscr{C}_c^\infty(G\times G, A\otimes_\pi A) \\ & a & \longmapsto & (g,h)\mapsto \mathrm{vol}(K)^{-1}\sum_{n\geq 0}\left[(a\cdot(\chi_n-\chi_{n-1}))(g)\otimes\chi_{n+1}(h)\right] \end{array}.$$

One can compute

$$\begin{aligned}(p_{\mathscr{A}}\circ s_{\mathscr{A}})(a)(gk) &= \mathrm{vol}(K)^{-1}\sum_{n\geq 0}\left(\int_K a(gk)(\chi_n-\chi_{n-1})(gk)\chi_{n+1}(gk)dk\right)\\ &= \mathrm{vol}(K)^{-1}\sum_{n\geq 0} a(gk)(\chi_n-\chi_{n-1})(gk)\cdot\int_K dk\\ &= a(gk)\cdot\mathrm{vol}(K)^{-1}\int_K dk = a(gk)\end{aligned}$$

because $\chi_{n+1}\equiv 1$ on $B(K,r_n)$. Now, choose a tubular neighborhood $\mathscr{N}\subseteq G\times G$ of $Y(K)\subseteq G\times G$ with projection $\pi:\mathscr{N}\twoheadrightarrow Y(K)$. Consider $\chi\in\mathscr{C}_c^\infty(\mathscr{N})$ such that $\chi\equiv 0$ near to $\partial\mathscr{N}$ and $\chi\equiv 1$ on $Y(K)\subseteq\mathscr{N}$. Define the linear map

$$\begin{array}{rccc} s_{\mathscr{B}}: & \mathscr{B} & \longrightarrow & F\otimes_\pi E = \mathscr{C}_c^\infty(G\times G, A\otimes_\pi A) \\ & b & \longmapsto & (g,h)\mapsto\begin{cases}\chi(g,h)b(\pi(g,h)) & \text{if } (g,h)\in\mathscr{N}\\ 0 & \text{otherwise}\end{cases} \end{array}.$$

A direct computation gives $(p_{\mathscr{B}}\circ s_{\mathscr{B}})(b)(g,gs)=\chi(g,gs)b(\pi(g,gs))=b(g,gs)$ because $(g,gs)\in Y(K)\subseteq\mathscr{N}$. In other words, $s_{\mathscr{A}}$ is a linear section of $p_{\mathscr{A}}$, $s_{\mathscr{B}}$ is a linear section of $p_{\mathscr{B}}$ and we showed that $\mathscr{C}_c^\infty(G/K,A)$ is quasi-Morita equivalent to $\mathscr{C}_c^\infty(Y(K),A)$ as expected.
2) Case $H=G$. Consider a smooth function with compact support $\chi'\in\mathscr{C}_c^\infty(G)$ such that $\int_G\chi'(g,g)dg=1$. Define the linear maps

$$\begin{array}{rccc} s_{\mathscr{A}}: & \mathscr{A} & \longrightarrow & E\otimes_\pi F \\ & a & \longmapsto & (g,h)\mapsto a\cdot\chi'(g,g)\end{array} \quad\text{and}\quad \begin{array}{rccc} s_{\mathscr{B}}: & \mathscr{B} & \longrightarrow & F\otimes_\pi E \\ & b & \longmapsto & (g,h)\mapsto b(g,h)\end{array}.$$

Again, direct computations give

$$(p_{\mathscr{A}}\circ s_{\mathscr{A}})(a)=\int_G a\cdot\chi'(gs,gs)ds = a\cdot\int_G\chi'(g,g)dg = a \text{ and } (p_{\mathscr{B}}\circ s_{\mathscr{B}})(b)(g,h)=b(g,h).$$

In other words, $s_{\mathscr{A}}$ is a linear section of $p_{\mathscr{A}}$, $s_{\mathscr{B}}$ is a linear section of $p_{\mathscr{B}}$ and we showed that A is quasi-Morita equivalent to $\mathscr{C}_c^\infty(Y(G),A)$. □

Theorem 7.3.4. *For every unital Fréchet algebra A endowed with a smooth action of G, there exists an isomorphism in periodic cyclic homology :*

$$HP_\bullet(\mathscr{C}_c^\infty(G/K,A)\rtimes G)\xrightarrow{\sim} HP_\bullet(A\rtimes K).$$

Démonstration. Since the algebras $\mathscr{C}_c^\infty(Y(K),A)$ and $\mathscr{C}_c^\infty(G/K,A)$ are G-equivariantly quasi-Morita equivalent and the algebras $\mathscr{C}_c^\infty(Y(G),A)$ and A are G-equivariantly (in particular K-equivariantly) quasi-Morita equivalent, the theorem 7.2.4 yields the following homotopy equivalences

$$\widehat{CC}(\mathscr{C}_c^\infty(G/K,A)\rtimes G)\xrightarrow{\sim}\widehat{CC}(\mathscr{C}_c^\infty(Y(K),A)\rtimes G)$$

$$\widehat{CC}(A\rtimes K)\xrightarrow{\sim}\widehat{CC}(\mathscr{C}_c^\infty(Y(G),A)\rtimes K).$$

But the algebras on the right are isomorphic under the composition

$$\mathscr{C}_c^\infty(Y(K),A)\rtimes G\xrightarrow[7.3.2]{\sim}\mathscr{C}_c^\infty(G,A)\rtimes_\alpha(G\times K)\underset{7.3.1}{\simeq}\mathscr{C}_c^\infty(G,A)\rtimes_\beta(G\times K)\xleftarrow[7.3.2]{\sim}\mathscr{C}_c^\infty(Y(G),A)\rtimes K.$$

The assertion follows. □

7.3.2 Smooth Clifford modules

Let us establish the following homotopy equivalence, it will be useful for the rest of the thesis.

Theorem 7.3.5. *Let M be a G-manifold and $\mathscr{A}$ be a Fréchet algebra on which G acts smoothly. The canonical inclusion $\mathscr{C}_c^\infty(M)\hookrightarrow\mathscr{C}_\tau^\infty(M)$ of smooth compactly-supported functions in the smooth Clifford module (see definition 2.2.7) induces a bounded chain homotopy equivalence*

$$\widehat{CC}((\mathscr{C}_c^\infty(M,\mathscr{A})\rtimes G))\xrightarrow{\sim}\widehat{CC}((\mathscr{C}_\tau^\infty(M)\otimes_\pi\mathscr{A})\rtimes G).$$

Démonstration. By definition, the Clifford multiplication $\xi\mapsto \mathrm{int}_{\xi^\sharp}-\mathrm{ext}_\xi$ extends to a morphism of algebras we denote $j:\mathrm{Cliff}_{\mathbb{C}}(T^\star M)\longrightarrow\mathrm{End}\big(\bigwedge T^\star M\big)$. We will write $i:\mathbb{C}\longrightarrow\mathrm{Cliff}_{\mathbb{C}}(T^\star M)$ for the canonical inclusion and $\mathrm{Tr}:\mathrm{End}(\bigwedge T^\star M)\longrightarrow\mathbb{C}$ for the trace map. We will show that i and $\mathrm{Tr}\circ j$ are inverse to each other modulo homotopy through the functor $\widehat{CC}((-\otimes_\pi\mathscr{A})\rtimes G)$. First, the composition :

$$\mathbb{C}\xrightarrow{i}\mathrm{Cliff}_{\mathbb{C}}(T^\star M)\xrightarrow{j}\mathrm{End}(T^\star M)\xrightarrow{\mathrm{Tr}}\mathbb{C}$$

sends $1\in\mathbb{C}$ to the trace of the identity matrix $id_{\mathrm{End}(\bigwedge T^\star M)}$, which is $\dim(\bigwedge T^\star M)=2^{\dim(M)}$. The composition $\mathrm{Tr}_\star\circ j_\star\circ i_\star$ is then homotopic to the identity at the level of algebra, and becomes chain-homotopic to the identity through the functor $\widehat{CC}((-\otimes_\pi\mathscr{A})\rtimes G)$.

Now we show the other side of the composition. Consider the unital algebras $R=\mathscr{C}^\infty(M)$ and $S=\Gamma^\infty(M,\mathrm{Cliff}_{\mathbb{C}}(T^\star M))$ and the morphism $i_\star:\mathscr{C}^\infty(M)\to\Gamma^\infty(M,\mathrm{Cliff}_{\mathbb{C}}(T^\star M))$. The following diagram commutes

$$\begin{array}{ccc}
\Gamma_c^\infty\big(M,\mathrm{End}(\bigwedge T^\star M)\big) & \xrightarrow{\mathrm{Tr}_\star} & \mathscr{C}_c^\infty(M)\\
{\scriptstyle (id_{\mathrm{End}}\otimes i)_\star}\downarrow & & \downarrow{\scriptstyle i_\star}\\
\Gamma_c^\infty\big(M,\mathrm{End}(\bigwedge T^\star M)\otimes\mathrm{Cliff}_{\mathbb{C}}(T^\star M)\big) & \xrightarrow{(\mathrm{Tr}\otimes id_{\mathrm{Cliff}})_\star} & \mathscr{C}_\tau^\infty(M)
\end{array}$$

since the bottom arrow is the tensorization by S along R of the top arrow. The composition $i_\star \circ \mathrm{Tr}_\star \circ j_\star$ can then be computed as

$$(\mathrm{Tr} \otimes id_{\mathrm{Cliff}})_\star \circ (id_{\mathrm{End}} \otimes i)_\star \circ j_\star : \mathscr{C}^\infty_\tau(M) \longrightarrow \mathscr{C}^\infty_\tau(M).$$

Moreover, the algebras R and S are $\mathbb{Z}/2\mathbb{Z}$-graded and the morphism $i_\star : R \to S$ preserves the grading. The diagram above then also commutes for the graded tensor product $\widehat{\otimes}$ instead of the ungraded tensor product $\otimes$. Consider the family of linear maps for $0 \le t \le \pi/2$

$$\begin{array}{cccc} h_t : & T^\star M & \longrightarrow & \mathrm{End}\big(\bigwedge T^\star M\big) \widehat{\otimes} \mathrm{Cliff}_{\mathbb{C}}(T^\star M) \\ & v & \longmapsto & \cos(t)\Big((\mathrm{int}_{v^\sharp} - \mathrm{ext}_v) \otimes id_{\mathrm{Cliff}}\Big) + \sin(t)\big(id_{\mathrm{End}} \otimes v\big) \end{array}.$$

For any $0 \le t \le \pi/2$, the square of h_t is equal to the multiplication by the scalar $\| v \|^2$ because the tensor product we considered is graded, hence extends to a morphism of algebras out of $\mathrm{Cliff}_{\mathbb{C}}(T^\star M)$. The family $\{h_t\}_{0 \le t \le \pi/2}$ then realizes a G-equivariant homotopy between

$$h_0 = 1_{\mathrm{End}} \otimes id_{\mathrm{Cliff}} : \mathrm{Cliff}_{\mathbb{C}}(T^\star M) \longrightarrow \mathrm{End}\big(\bigwedge T^\star M\big) \widehat{\otimes} \mathrm{Cliff}_{\mathbb{C}}(T^\star M)$$

$$h_1 = (id_{\mathrm{End}} \otimes i) \circ j : \mathrm{Cliff}_{\mathbb{C}}(T^\star M) \longrightarrow \mathrm{End}\big(\bigwedge T^\star M\big) \widehat{\otimes} \mathrm{Cliff}_{\mathbb{C}}(T^\star M).$$

We showed that $i_\star \circ \mathrm{Tr}_\star \circ j_\star : \mathscr{C}^\infty_\tau(M) \longrightarrow \mathscr{C}^\infty_\tau(M)$ is homotopic to the composition $(\mathrm{Tr} \otimes id_{\mathrm{Cliff}})_\star \circ (1_{\mathrm{End}} \otimes id_{\mathrm{Cliff}})_\star = (\mathrm{Tr}(1_{\mathrm{End}}) \otimes id_{\mathrm{Cliff}})_\star$ which is noting other than the multiple of the identity by the scalar $2^{\dim(M)}$. It makes the composition $i \circ \mathrm{Tr} \circ j$ chain-homotopic to the identity at the level of periodic cyclic complexes through the functor $\widehat{CC}((- \otimes_\pi \mathscr{A}) \rtimes G)$, which ends up the proof. $\square$

8 Periodic cyclic homology of crossed product algebras

Sommaire

Let G be a connected real reductive Lie group, K a maximal compact subgroup of it and A a complete locally convex algebras endowed with a smooth action of G as before. The aim of this chapter is to show that the Dirac element $\alpha \in K_G^{\dim(G/K)}(\mathscr{C}_\tau(G/K))$ induces an isomorphism of periodic cyclic homology groups

$$(\alpha^\sharp \otimes id_{\mathscr{A}}) \rtimes G : HP_\bullet(\mathscr{C}_c^\infty(G/K, \mathscr{A}) \rtimes G) \xrightarrow{\sim} HP_{\bullet + \dim(G/K)}(\mathscr{A} \rtimes G).$$

where $\mathscr{A} = A \otimes_\pi \mathscr{D}$ is the stabilization of A with a certain Schatten ideal $\mathscr{D}$. To obtain this isomorphism, we will *descend* the construction of section §5 from bivariant K-theory to periodic cyclic homology by refining the Cuntz identification 5.1.5 for the Dirac and dual Dirac elements. We will show that they associated Fréchet algebra homomorphisms are K-equivariantly inverses to each other at the level of periodic cyclic complexes. The passage form K-equivariance to G-equivariance is a consequence of the theorem 1.2.4 of Nistor [Nis93] which identifies periodic cyclic homology of crossed product algebras by G and K on each conjugacy classes, the Dirac element being define globally. The main reference for descent from bivariant K-theory to periodic cyclic homology are [Nis93] and [Nis91].

8.1 Banach version of the Cuntz algebra

In this section, we adapt the Cuntz algebra qA defined in 5.1.3 into a rescaling family of algebras $\{q_R A\}_{R>0}$ playing an analogous role for bounded homomorphisms of Fréchet algebras instead of $C^\star$-homomorphisms.

8.1.0.1 The Banach free product

Let A be a Banach algebra. Recall the construction of the algebraic free product in section §5 as the coproduct in the category of complex algebras. In order to work in a Banach framework, it is necessary to equip the algebraic free product $A \star A$ with a submultiplicative norm compatible with given norms on the algebras. Given $R \geq 1$, we define the norm $\|\cdot\|_R$ to be the largest norm which coincide on each factor $A^{\otimes_\pi n}$ with nR^{n-1} times the projective crossed norm. This sub-multiplicative norm may be thought as a rescaled version of the projective norm.

Definition 8.1.1. *Given a Banach algebra A and $R \geq 1$, we define the* ***Banach R-free product*** *$Q_R A$ to be the completion of the algebraic free product $A \star A$ with respect to the norm $\|\cdot\|_R$.*

The algebra $Q_R A$ is a rescaled version of the classical $C^\star$-free product QA defined in 5.1.2 and allows to control a larger class of homomorphisms than just the contractive ones, as shows the next lemma.

Lemma 8.1.2. *For any pair of Banach algebra homomorphisms $\phi_0, \phi_1 : A \to B$ of norm at most R there exists a unique homomorphism $\phi_0 \star \phi_1 : Q_R A \to B$ verifying :*

$$(\phi_0 \star \phi_1) \circ \iota_0 = \phi_0,\ (\phi_0 \star \phi_1) \circ \iota_1 = \phi_1 \textit{ and } \|\phi_0 \star \phi_1\| \leq R.$$

It is given by $(\phi_0 \star \phi_1)(a_0 \cdots a_n) = \phi_0(a_0)\phi_1(a_1)\phi_0(a_2)\cdots\phi_{n \bmod 2}(a_n)$ on the reduced words.

Démonstration. It is clear that the definition of $\phi_0 \star \phi_1$ on the reduced words extends to a well-defined homomorphism on the whole algebra $Q_R A$. One can easily check the computations $(\phi_0 \star \phi_1) \circ \iota_0 = \phi_0$ and $(\phi_0 \star \phi_1) \circ \iota_1 = \phi_1$. It remains to show that $\phi_0 \star \phi_1$ is of norm at most R with respect to $\|\cdot\|_R$. Taking $x = \sum_{i \in I} a_0^{(i)} \cdots a_{n_i}^{(i)} \in A \star A$ we compute :

$$\begin{aligned}
\|(\phi_0 \star \phi_1)(x)\|_B &\leq \sum_{i \in I} \|(\phi_0 \star \phi_1)(a_0^{(i)} \cdots a_{n_i}^{(i)})\|_B. \\
&= \sum_{i \in I} \|\phi_0(a_0^{(i)})\|_B \|\phi_1(a_1^{(i)})\|_B \cdots \|\phi_{n_i \bmod 2}(a_{n_i}^{(i)})\|_B \\
&\leq \sum_{i \in I} R^{n_i+1} \|a_0^{(i)}\|_A \cdots \|a_{n_i}^{(i)}\|_A && \text{(because } \|\phi_0\|, \|\phi_1\| \leq R) \\
&\leq R\left(\sum_{i \in I} (n_i+1) R^{n_i} \|a_0^{(i)}\|_A \cdots \|a_{n_i}^{(i)}\|_A\right) && \text{(because } n_i \geq 1) \\
&\leq R\|x\|_R.
\end{aligned}$$

It ends up the proof. ☐

Lemma 8.1.3. *Any bounded algebra morphism $\phi : A \to B$ induces a morphism of between the respective free algebras $Q\phi : Q_{R'} A \longrightarrow Q_R B$ when $R' \geq R\|\phi\|$.*

Démonstration. Composing ϕ with the canonical inclusions $\iota_0, \iota_1 : B \to Q_R B$ leads to a pair of algebra homomorphisms $(\iota_0 \circ \phi), (\iota_1 \circ \phi) : A \longrightarrow Q_R B$. It induces an algebra homomorphism $Q_{R'} A \to Q_R B$ for all R' greater to the norm of both $\iota_0 \circ \phi$ and $\iota_1 \circ \phi$. We compute $\|(\iota_0 \circ \phi)(a)\|_R = \|\phi(a)\|_R \leq R\|\phi\| \times \|a\|_A$. Same thing occurs for $\iota_1 \circ \phi$, which gives the expected result. □

The above lemma shows that the construction $A \rightsquigarrow Q_R A$ is functorial with respect to bounded algebra homomorphisms, provided adjustment of the parameter R. This flexibility will be crucial for the following constructions.

The two inclusions on first and second coordinates $A \rightrightarrows A \oplus A$ are isometries. By definition of the free product, they induce for all $R \geq 1$ an algebra homomorphism we denote $i : Q_R A \to A \oplus A$ which verifies for all $a \in A$ the identities $i(\iota_0(a)) = (a, 0)$ and $i(\iota_1(a)) = (0, a)$. The following theorem is proved in [Cun87] for the $C^\star$-algebraic setting and we adapted the proof for the Banach framework.

Theorem 8.1.4. *For all $R \geq 1$, the algebra homomorphism $i : Q_R A \to A \oplus A$ is homotopic to the identity modulo 2×2-square matrices. Moreover, if A is a G-algebra, the homotopy equivalence is G-equivariant.*

Remark Before to show the theorem, we need to fix a topology on the space of square matrices. If B is a Banach algebra, we endow $M_2(B)$ with the topology induced by the multiplicative Hilbert-Schmidt norm. With this norm the space of square matrices with coefficients in B is Banach and the matrices

$$u_t := \begin{pmatrix} \cos\left(\frac{\pi}{2} t\right) & \sin\left(\frac{\pi}{2} t\right) \\ -\sin\left(\frac{\pi}{2} t\right) & \cos\left(\frac{\pi}{2} t\right) \end{pmatrix} \tag{35}$$

are isometries for all $t \in [0, 1]$.

Démonstration. Let us take the algebra homomorphism

$$\begin{array}{rccc} j : & A \oplus A & \longrightarrow & M_2(Q_R A) \\ & (a, b) & \longmapsto & \begin{pmatrix} \iota_0(a) & 0 \\ 0 & \iota_1(b) \end{pmatrix} \end{array}.$$

We will prove that it defines an inverse to i modulo smooth homotopy. First, let us consider the composition $M_2(i) \circ j : A \oplus A \to M_2(A \oplus A)$. We want to show that it is homotopic to the top-left corner inclusion of $A \oplus A$ in its space of square matrices. We compute for $(a, b) \in A \oplus A$:

$$(M_2(i) \circ j)(a, b) = \begin{pmatrix} i(\iota_0(a)) & 0 \\ 0 & i(\iota_1(b)) \end{pmatrix} = \begin{pmatrix} (a, 0) & 0 \\ 0 & (0, b) \end{pmatrix}.$$

The continuous family of homomorphisms

$$\begin{array}{rccc} w(t): & A\oplus A & \longrightarrow & M_2(A\oplus A) \\ & (a,b) & \longmapsto & \begin{pmatrix} (a,0) & 0 \\ 0 & 0 \end{pmatrix} u_t \begin{pmatrix} 0 & 0 \\ 0 & (0,b) \end{pmatrix} u_t^{-1} \end{array}$$

where u_t is the rotation matrix (35) realizes a smooth homotopy between $w(0) = M_2(i)\circ j$ and the top-left corner inclusion $w(1) = A\oplus A \hookrightarrow M_2(A\oplus A)$.
Now consider the other side composition $j\circ i : Q_R A \to M_2(Q_R A)$. We want to check that it is smoothly homotopic to the top-left corner inclusion of $Q_R A$ in its space of square matrices. For all $a\in A$, the morphism $j\circ i$ verifies, by definition of i as a free product, the following identities :

$$(j\circ i)(\iota_0(a)) = j(a,0) = \begin{pmatrix} \iota_0(a) & 0 \\ 0 & 0 \end{pmatrix} \text{ and } (j\circ i)(\iota_1(a)) = j(a,0) = \begin{pmatrix} 0 & 0 \\ 0 & \iota_1(a) \end{pmatrix}.$$

Since u_t is an isometry for the Hilbert-Schmidt norm for all $t\in[0,1]$, the following pair of isometric homomorphisms

$$\begin{array}{rccc} & A & \longrightarrow & M_2(Q_R A) \\ \gamma_0 : & a & \longmapsto & \begin{pmatrix} \iota_0(a) & 0 \\ 0 & 0 \end{pmatrix} \\ \gamma_1(t): & a & \longmapsto & u_t \begin{pmatrix} 0 & 0 \\ 0 & \iota_1(a) \end{pmatrix} u_t^{-1} \end{array}$$

induces for all $R\geq 1$ an algebra homomorphism $\gamma_0\star\gamma_1(t) : Q_R A \to M_2(Q_R A)$. When $t=0$ we recover exactly $j\circ i$ by the computation above and when $t=1$ we get the top-left corner inclusion $Q_R A \hookrightarrow M_2(Q_R A)$. We proved that the algebra homomorphism i is smoothly homotopic to identity modulo square matrices, which ends up the proof. □

8.1.0.2 The Banach Cuntz' algebra

Definition 8.1.5. *For all $R\geq 1$, the **Banach Cuntz's algebra** $q_R A$ associated to A is defined to be the kernel of the fold map $q_R A := \ker(id\star id : Q_R A \to A)$.*

By definition of $q_R A$, the following exact sequence is double-split for all A and $R>0$:

$$0 \longrightarrow q_R A \hookrightarrow Q_R A \underset{\curvearrowleft\ \iota_0,\,\iota_1}{\xrightarrow{id\star id}} A \longrightarrow 0\ . \tag{36}$$

Again, this algebra is a Banach version of the Cuntz' algebra qA we defined in 5.1.3.

Lemma 8.1.6. *The algebra $q_R A$ is the two-side ideal of $Q_R A$, generated by the differences $\iota_0(a)-\iota_1(a)$ for $a\in A$. In particular, any element $x = \iota_0(a_0)\iota_1(a_1)\cdots\iota_{n\ mod\ 2}(a_n)\in q_R A$*

with $a_i \in A$ can be written as :

$$x = \sum_{k=0}^{\lfloor n/2 \rfloor} \iota_1(a_0 \cdots a_{2k-1})\Big(\iota_0(a_{2k}) - \iota_1(a_{2k})\Big)\iota_1(a_{2k+1})\iota_0(a_{2k+1}) \cdots \iota_{n\, mod\, 2}(a_n).$$

Démonstration. It is clear that any difference $\iota_0(a) - \iota_1(a)$ with $a \in A$ belongs to $q_R A$, *i.e.* to the kernel of the fold map because $(id \star id)(\iota_0(a)) = a = (id \star id)(\iota_1(a))$ by the universal property of the free product $Q_R A$. Then $q_R A$ contains the two-side ideal generated by the differences. On the other side we need to show that any element $x = \iota_0(a_0)\iota_1(a_1) \cdots \iota_{n \bmod 2}(a_n) \in q_R A$ lies in this ideal. The telescopic sum above is an element of this ideal and a straightforward computation shows that it is equal to $x - \iota_1(a_0 \cdots a_n) \in Q_R A$. But because x is in the kernel of the fold map,

$$0 = (id \star id)(x) = (id \star id)(\iota_0(a_0)\iota_1(a_1) \cdots \iota_{n \bmod 2}(a_n)) = a_0 \cdots a_n \in A.$$

Then the sum above is exactly equal to x and we showed the lemma. □

Lemma 8.1.7. *For every integer $N \geq 0$, the natural inclusion $j : (q_R A)^N \hookrightarrow q_R A$ induces a chain-homotopy equivalence of periodic cyclic complexes.*

Démonstration. There exists natural isomorphism of vector spaces :

$$\begin{array}{ccc} Q_R A & \simeq & \Omega A \\ \iota_0(a_0) q(a_1) \cdots q(a_n) & \leftrightarrow & a_0 da_1 \cdots da_n \\ q(a_1) \cdots q(a_n) & \leftrightarrow & da_1 \cdots da_n \end{array}$$

where $q(a)$ denotes the difference $\iota_0(a) - \iota_1(a)$ for any $a \in A$. Under this isomorphism, the $q_R A$-adic filtration of $Q_R A$ corresponds to the degree filtration on ΩA, which is $Q_R A/(q_R A)^N = \Omega^{<n} A$. This identification yields to canonical linear sections of the algebra epimorphisms for any $N \geq 0$:

$$Q_R A \to Q_R A/(q_R A)^N, \text{ and } Q_R A/(q_R A)^N \longrightarrow Q_R A/q_R A = A.$$

In other words, we obtained two split algebras extensions where all vertical arrows have also bounded linear sections :

$$\begin{tikzcd} 0 \arrow[r] & (q_R A)^N \arrow[r] \arrow[d, "j"'] & Q_R A \arrow[r] \arrow[d] & Q_R A/(q_R A)^N \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & q_R A \arrow[r] & Q_R A \arrow[r] & Q_R A/q_R A \arrow[r] & 0 \end{tikzcd}$$

The extensions give rise via the excision theorem in periodic cyclic homology 4.3.3 to long exact sequences of homology groups. As $Q_R A/(q_R A)^n \to Q_R A/q_R A$ has nilpotent kernel, it induces a chain-homotopy equivalence of periodic cyclic complexes by Goodwillie's Theorem [Goo85]. Then, the right vertical arrow and the central vertical arrow are chain-homotopy equivalences which gives that $j : \widehat{CC}\big((q_R A)^N\big) \longrightarrow \widehat{CC}(q_R A)$ is also a chain-homotopy equivalence as expected. □

The following theorem is central in the study. It relies on the fact that the homotopy equivalence between the free product $Q_R A$ and $A \oplus A$ is enough stable to be preserved under smooth actions of group and tensorization. Again, it is a Fréchet and equivariant version of the constructions of [Cun87].

Theorem 8.1.8. *If A and B are G-algebras, the canonical algebra homomorphism $id \star 0 : q_R A \to A$ induces a chain homotopy equivalence through the functor $\widehat{CC}((- \otimes_\pi B) \rtimes G)$, which is*

$$\big((id \star 0) \otimes_\pi id_B\big) \rtimes G : \widehat{CC}\Big((q_R A \otimes_\pi B) \rtimes G\Big) \xrightarrow{\sim} \widehat{CC}\Big((A \otimes_\pi B) \rtimes G\Big).$$

Démonstration. Tensoring the double-split extension $0 \to q_R A \to Q_R A \to A \to 0$ (see equation (36)) with B, and then taking the G-crossed product yields the extension

$$0 \longrightarrow (q_R A \otimes_\pi B) \rtimes G \longrightarrow (Q_R A \otimes_\pi B) \rtimes G \longrightarrow (A \otimes_\pi B) \rtimes G \longrightarrow 0$$

which is doubly-split via the linear maps $(\iota_0 \otimes id_B) \rtimes G$ and $(\iota_1 \otimes id_B) \rtimes G$. The argument of Cuntz 8.1.4 showing that $Q_R A \to A \oplus A$ is a stable homotopy equivalence also applies to $(Q_R A \otimes_\pi B) \rtimes G \longrightarrow (A \otimes_\pi B) \rtimes G \oplus (A \otimes_\pi B) \rtimes G$, which implies the expected result. □

Corollary 8.1.9. *For any G-algebras A, A', B and $R > 0$, the morphisms :*

$$\Delta_R = \Big(\big(\iota_0^A \otimes id\big) \star \big(\iota_1^A \otimes id\big)\Big)_{|q_R} : q_R(A \otimes_\pi A') \longrightarrow q_R A \otimes_\pi A',$$

$$\nabla_R = \Big(\Big(id \otimes \iota_0^{A'}\Big) \star \Big(id \otimes \iota_1^{A'}\Big)\Big)_{|q_R} : q_R(A \otimes_\pi A') \longrightarrow A \otimes_\pi q_R A'$$

induce chain-homotopy equivalences trough the functor $\widehat{CC}((- \otimes_\pi B) \rtimes G)$.

Démonstration. It is a consequence of the identities

$$((id_A \star 0) \otimes id_{A'}) \circ \Delta_R = id_{A \otimes_\pi A'} \star 0 \quad \text{and} \quad (id_A \otimes (id_{A'} \star 0)) \circ \nabla_R = id_{A \otimes_\pi A'} \star 0$$

and the fact that $id \star 0$ is a chain-homotopy equivalences through the functor $\widehat{CC}((- \otimes_\pi B) \rtimes G)$ due to the theorem 8.1.8 above. □

Remark Given a Fréchet algebra A and a derivation D over it, we can equip A with the $\mathscr{C}^k$-sup semi-norms $\|a\|_k := \|D^k(a)\|$ controlling the growth of the derivatives. This structure naturally endow the Cuntz algebra $q_R A$ with a family of semi-norms

$$\|\iota_1(a_1) \cdots \iota_{n \bmod 2}(a_n)\|_{(R,m)} := \sum_{\substack{(m_1, \cdots, m_n) \in \mathbb{N}^n \\ m_1 + \cdots m_n \leq m}} R^n \|a_1\|_{m_1} \cdots \|a_n\|_{m_n}. \tag{37}$$

For a fixed R, we will denote by $q_R^\infty A$ the completion of $q_R A$ for the norms $\|\cdot\|_{(R,m)}$, where $m \in \mathbb{N}$. All the result established above for $q_R A$ also hold for $q_R^\infty A$.

8.2 Dirac-dual Dirac method in periodic cyclic homology

Let G be a connected Lie-group and $K \subset G$ a maximal compact subgroup of it. The group G acts by left translation on the quotient space G/K. **We make the assumption that G/K is even-dimensional**. We fix over G/K a G-invariant Riemannian metric. Recall the definitions of the Dirac element $\alpha \in K^0_G(\mathscr{C}_\tau(G/K))$ and the dual Dirac element $\beta \in K_0^K(\mathscr{C}_\tau(G/K))$ (see §3.3.3 and 2.2.9) over the complete manifold G/K :

$$\alpha := \left(\mathscr{H} = \mathscr{L}^2(\Omega_c^\star(G/K)),\ \rho_{d+d^\star},\ F = \frac{d+d^\star}{(1+\Delta)^{1/2}},\ U = \text{left transl.}\right),$$

$$\beta := [e_0] - [e_1].$$

The Riemannian connection ∇ acts as a derivation on the smooth Clifford module $\mathscr{C}_\tau^\infty(G/K)$ defined in 2.2.7, which is the space of smooth sections with compact support on the Clifford bundle. It is endowed with a Fréchet topology coming from the family of $\mathscr{C}^k$-sup norms $\|\cdot\|_k$ with $k \in \mathbb{N}$ we defined in the remark above. Hence, we equip $q_R^\infty(\mathscr{C}_\tau^\infty(G/K))$ with the family of norms $\|\cdot\|_{(R,m)}$ with $R > 0$ and $m \in \mathbb{N}$ we defined in (37). In this subsection, we verify that in the Dirac and dual Dirac elements descend to morphisms :

$$\alpha^\sharp : q_R^\infty(\mathscr{C}_\tau^\infty(G/K)) \longrightarrow \ell^{p,G}(\mathscr{H}) \text{ and } \beta^\sharp : q_1\mathbb{C} \longrightarrow \mathscr{C}_\tau^\infty(G/K) \otimes_\pi M_2(\mathbb{C})$$

for R great enough where we recall that $\ell^{p,G}(\mathscr{H})$ is the G-smooth p-th Schatten ideal (definition 5.1.7). We also compute their composition $\alpha^\sharp \circ q_R^\infty(\beta^\sharp)$ which turns out to be K-equivariantly homotopic to the morphism representing the trivial class $\mathbb{1}_K \in KK_K(\mathbb{C},\mathbb{C})$ (see 5.1.8). In order words, this subsection refine the Cuntz identification at the level of Fréchet algebras for the Dirac and dual Dirac elements and compute a one-side composition using Kasparov theorem 5.2.1.

Proposition 8.2.1. *The Dirac element $\alpha \in K^0_G(\mathscr{C}_\tau(G/K))$ can be represented by a G-equivariant Kasparov bimodule $(\mathscr{H},\rho,F)$ such that for $p > \dim(G/K)$ there exist a constant $R > 0$ verifying for all $f \in \mathscr{C}_\tau(G/K)$:*

$$F^2 = 1, \quad \|\rho(f)\| \le R\|f\|_0 \ \textit{and} \ \|[F,\rho(f)]\| \le R\|f\|_1.$$

Démonstration. According to Connes [Con85], pages 89-91 one may obtain, after adding a degenerate Kasparov bimodule to α, a G-invariant Fredholm module $(\mathscr{H},\rho,F)$ which represents the same class as α in the G-equivariant KK-group $KK_G(C_\tau(G/K),\mathbb{C})$, is p-summable over $\mathscr{C}_c^\infty(G/K)$ for $p > dim(G/K)$ and satisfies $F^2 = Id$. It is clear that the morphism ρ is bounded because the Clifford multiplication representing α was already bounded. We have to show that the commutators $[F,\rho(f)]$ is also for the norm ℓ^p, when $p > \dim(G/K)$. Let $x \in G/K$ and $B_1 \subset B_2 \subset B_3 \subset G/K$ be balls of center x and strictly increasing radii. We will show that $[F,\rho(f)]$ is ℓ^p-bounded for any $f \in \mathscr{C}_c^\infty(B_1)$. Fix a real cutoff function $\chi \in \mathscr{C}_c^\infty(B_2)$ which is constant equal to 1 on B_1

and $\mu \in \mathscr{C}_c^\infty(B_3)$ which is constant equal to 1 on B_2. The commutator becomes :

$$[F, \rho(f)] = [F, \rho(f)]\chi + \rho(f)[F, \chi].$$

The image of the operator $\rho(f)[F, \chi]$ consists of functions supported in B_1. Viewing them as functions on $S^n \approx G/K \cup \{\infty\}$ vanishing outside B_1 we may interprete $\rho(f)[F, \chi]$ as pseudodifferential operator from G/K to S^n. Its order is -1 so that it extends to a bounded operator $\rho(f)[F, \chi] : \mathscr{L}^2(G/K) \longrightarrow \mathscr{H}^{(1)}(S^n)$ whose norm is majorized in terms of f, χ and their first derivatives, i.e. in terms of $\|f\|_1$. Now we may factorize

$$\rho(f)[F, \chi] : \mathscr{L}^2(G/K) \longrightarrow \mathscr{H}^{(1)}(S^n) \longrightarrow \mathscr{L}^2(S^n) \xrightarrow{\cdot\mu} \mathscr{L}^2(G/K)$$

where the last map is given by multiplication with μ, followed by extension by 0 outside B_3. The first and last maps are bounded operators whereas the middle one lies in the p-th Schatten ideal for p great enough. The assertion for $\rho(f)[F, \chi]$ follows because ℓ^p is a two-sided ideal in $\mathscr{L}(H)$. Consider now the bounded operator $[F, \rho(f)]\chi$ on $\mathscr{L}^2(G/K)$. The operator F is bounded and self-adjoint on $\mathscr{L}^2(G/K)$. Now a bounded operator on Hilbert space belongs to the Schatten ideal ℓ^p if and only if its adjoint does and the Schatten norms of the two operators coincide. We find

$$([F, \rho(f)]\chi)^\star = -\chi[F, \rho(f)]$$

and our previous arguments apply to the latter operator. This proves that $[F, \rho(f)]$ is bounded for the ℓ^p-norm. Choose R great enough with respect to the norm of ρ and of the commutator $[F, \rho(f)]$. □

Since G/K is supposed to be even-dimensional, write the underlying Hilbert space as a direct sum $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ and denote P_+ for the projection to the even part. Consider the morphisms

$$\rho_0 := P_+\rho P_+ : \mathscr{C}_\tau^\infty(G/K) \longrightarrow \mathscr{L}(\mathscr{H}) \text{ and } \rho_1 := P_+F\rho(f)FP_+ : \mathscr{C}_\tau^\infty(G/K) \longrightarrow \mathscr{L}(\mathscr{H}).$$

Theorem 8.2.2. *The Dirac element α induces, for $p > \dim(G/K)$ and $R > 0$ great enough, a bounded algebra homomorphism :*

$$\alpha^\sharp = (\rho_0 \star \rho_1)_{|q_R^\infty} : q_R^\infty(\mathscr{C}_\tau^\infty(G/K)) \longrightarrow \ell^{p,G}(\mathscr{H}).$$

where we recall that $\ell^{p,G}(\mathscr{H})$ is the G-smooth p-th Schatten ideal (definition 5.1.7).

Démonstration. Take R as in the lemma above. Due to the topology in $q_R^\infty(\mathscr{C}_\tau^\infty(G/K))$ coming from the $\mathscr{C}^k$-sup norms on $\mathscr{C}_\tau^\infty(G/K)$, it suffices to show that for any element $x = \iota_0(f_0)\iota_1(f_1)\cdots\iota_{n \bmod 2}(f_n) \in q_R^\infty(\mathscr{C}_\tau^\infty(G/K))$, it exists a constant $C > 0$ and an integer $m \in \mathbb{N}$ such that

$$\|(\rho_0 \star \rho_1)(x)\|_{\ell^p(H)} \leq C\|x\|_{(R,m)}$$

where the norm $\|\cdot\|_{(R,m)}$ have been defined in (37). Due to the lemma 8.1.6, we can write

$$(\rho_0 \star \rho_1)(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \rho_1(f_0 \cdots f_{2k-1})\Big(\rho_0(f_{2k}) - \rho_1(f_{2k})\Big)\rho_1(f_{2k+1}) \cdots \rho_{n \bmod 2}(f_n).$$

The free product $\rho_0 \star \rho_1$ applied to $x \in q_R A$ is of norm :

$$\begin{aligned}
\|(\rho_0 \star \rho_0)(x)\|_{\ell^p(H)} &\leq \sum_{k=0}^{\lfloor n/2 \rfloor} \|\rho_1(f_0 \cdots f_{2k-1})\Big(\rho_0(f_{2k}) - \rho_1(f_{2k})\Big)\rho_1(f_{2k+1}) \cdots \rho_{n \bmod 2}(f_n)\|_{\ell^p(H)} \\
&\leq \sum_{k=0}^{\lfloor n/2 \rfloor} \|\rho_1(f_0 \cdots f_{2k-1})\|_{L(H)} \|\rho_0(f_{2k}) - \rho_1(f_{2k})\|_{\ell^p(H)} \|\rho_1(f_{2k+1}) \cdots \rho_{n \bmod 2}(f_n)\|_{L(H)} \\
&\leq \sum_{k=0}^{\lfloor n/2 \rfloor} R^n \|f_0\|_0 \cdots \|f_{2k-1}\|_0 \|\rho_0(f_{2k}) - \rho_1(f_{2k})\|_{\ell^p(H)} \|f_{2k+1}\|_0 \cdots \|f_n\|_0 && \text{(by 8.2.1)} \\
&\leq \sum_{k=0}^{\lfloor n/2 \rfloor} R^{n+1} \|f_0\|_0 \cdots \|f_{2k-1}\|_0 \|f_{2k}\|_1 \|f_{2k+1}\|_0 \cdots \|f_n\|_0 && \text{(by 8.2.1)} \\
&\leq R\|x\|_{(R,1)},
\end{aligned}$$

which ends up the proof of boundness. Finally, the image of $\rho_0 \star \rho_1$ is contained in the space of smooth vectors of the G-module $\ell^p(\mathcal{H})$ because F is G-equivariant and acts smoothly on $\mathcal{C}_\tau^\infty(G/K)$. □

Proposition 8.2.3. *The dual Dirac element $\beta = [e_0] - [e_1] \in K_0^K(\mathcal{C}_\tau(G/K))$ induces a K-equivariant bounded algebra homomorphism*

$$\beta^\sharp = (e_0 \star e_1)_{|q_1\mathbb{C}} : q_1\mathbb{C} \longrightarrow \mathcal{C}_\tau^\infty(G/K) \otimes_\pi M_2(\mathbb{C}).$$

Démonstration. It is a consequence of the universal property 8.1.2 of the Banach free algebras defined before because the idempotent 2×2-matrices e_0 and e_1 are both of norm 1. □

Proposition 8.2.4. *The following algebra homomorphisms are K-equivariantly homotopy equivalent, where we recall $\mathbb{1}_K$ is defined in 5.1.8 :*

$$\alpha^\sharp \circ q_R^\infty(\beta^\sharp) : q_R(q_1\mathbb{C}) \longrightarrow M_2(\ell^{p,G}(\mathcal{H})) \;\; \textit{and} \;\; \mathbb{1}_K : q_R(q_1\mathbb{C}) \longrightarrow M_2(\ell^{p,G}(\mathcal{H})).$$

Démonstration. This assertion corresponds to the restriction of the corollary 5.2.3 to $q_R(q_1\mathbb{C}) \subseteq q(q\mathbb{C})$. □

8.3 Application to crossed product algebras

Now we have defined the Dirac and dual Dirac in a Fréchet framework and verify their one-side composition property, we will be interested in considering crossed product algebras. Fix a complete locally convex algebra A on which the Lie group G acts smoothly as before.

8.3.1 The stabilized algebra

Definition 8.3.1. *The **stabilization algebra** is defined to be the Fréchet G-algebra $\mathscr{D} = \ell^{p,G}(\mathscr{H}')$ of smooth elements in the p-th Schatten ideal associated to the Hibert space $\mathscr{H}' = \bigoplus_{\mathbb{N}} L^2(G, d\mu)$ made of a the direct sum of countably many copies of the regular representation of G. Will use the notation $\mathscr{A} = A \otimes_\pi \mathscr{D}$ for the **stabilized algebra** of A.*

Remark Given any Hilbert space $\mathscr{H}$, there exists a *non-canonical* isomorphism $\ell^{p,G}(\mathscr{H}) \otimes_\pi \ell^{p,G}(\mathscr{H}') \simeq \ell^{p,G}(\mathscr{H} \otimes \mathscr{H}') \simeq \ell^{p,G}(\mathscr{H}')$, which makes the stabilized algebra *stable* under tensorization by smooth Schatten spaces, *i.e.* it exists a G-equivariant isomorphism

$$\ell^{p,G}(\mathscr{H}) \otimes_\pi \mathscr{A} \simeq \mathscr{A}. \tag{38}$$

Lemma 8.3.2. *For every G-algebra A with stabilization $\mathscr{A}$, the following chain-complex homomorphism is an chain-homotopy equivalence*

$$((1 \mapsto p_1) \otimes id_{\mathscr{A}}) \rtimes G : \widehat{CC}(\mathscr{A} \rtimes G) \longrightarrow \widehat{CC}\Big(\big(M_2(\ell^{p,G}(\mathscr{H})) \otimes_\pi \mathscr{A}\big) \rtimes G\Big).$$

Démonstration. The G-equivariant isometric isomorphism of Hilbert spaces

$$\begin{array}{cccc} \Psi : & \mathscr{H} \widehat{\otimes}_\pi \mathscr{H}' & \longrightarrow & \mathscr{H}^{\mathrm{triv}} \widehat{\otimes}_\pi \mathscr{H}' \\ & \xi \otimes f & \longmapsto & (g \mapsto \pi(g)(\xi) \cdot f(g)) \end{array}$$

induces a G-equivariant isomorphism at the level of the p-th Schatten ideals :

$$\Psi' : \ell^{p,G}(\mathscr{H}) \otimes_\pi \mathscr{D} \underset{\sim}{\overset{\mathrm{Ad}(\Psi)}{\longrightarrow}} \ell^{p,G}(\mathscr{H}^{\mathrm{triv}}) \otimes_\pi \mathscr{D} = \ell^{p}(\mathscr{H}^{\mathrm{triv}}) \otimes_\pi \mathscr{D}.$$

The last equality comes from the fact that the action of G being trivial, any vector in the Schatten ideal is smooth with respect to it. In particular, $\ell^p(\mathscr{H}^{\mathrm{triv}}) \otimes_\pi \mathscr{A} \simeq \ell^{p,G}(\mathscr{H}) \otimes_\pi \mathscr{A}$ in a G-equivariant wy and the chain-complex morphism of the statement descends to :

$$((1 \mapsto p_1) \otimes id_{\mathscr{A}}) \rtimes G : \widehat{CC}(\mathscr{A} \rtimes G) \longrightarrow \widehat{CC}\Big(\big(M_2(\ell^{p}(\mathscr{H}^{\mathrm{triv}})) \otimes_\pi \mathscr{A}\big) \rtimes G\Big).$$

We want to show that it defines chain-homotopy equivalence. In [Sim10], he states that the canonical homomorphism $\mathscr{H}^{\mathrm{triv}} \otimes_\pi (\mathscr{H}^{\mathrm{triv}})^\vee \longrightarrow \ell^1(\mathscr{H}^{\mathrm{triv}})$ is an isomorphism of Banach algebras. The vector spaces $E = \mathscr{H}^{\mathrm{triv}} \otimes_\pi \mathscr{A}$ and $F = (\mathscr{H}^{\mathrm{triv}})^\vee$ then define a G-equivariant quasi-Morita datum between $\mathscr{A}$ and $\ell^1(\mathscr{H}^{\mathrm{triv}}) \otimes_\pi \mathscr{A}$ according to the lemma 7.2.2. Moreover, the equivariant quasi-Morita equivalence is realized sending $1 \in \mathbb{C}$ to the idempotent operator p_1 of rank one :

$$(1 \mapsto p_1) \otimes id_{\mathscr{A}} : \mathscr{A} \longrightarrow \ell^1(\mathscr{H}^{\mathrm{triv}}) \otimes_\pi \mathscr{A}.$$

The case $p = 1$ follows from theorem 7.2.4 and Morita invariance of periodic cyclic homology 4.3.3 (to make disappear the matrix algebra) :

$$((1 \mapsto p_1) \otimes id_{\mathscr{A}}) \rtimes G : \widehat{CC}(\mathscr{A} \rtimes G) \longrightarrow \widehat{CC}\Big((M_2(\ell^1(\mathscr{H}^{\text{triv}})) \otimes_\pi \mathscr{A}) \rtimes G\Big).$$

Write $\mathscr{A}_p := (\ell^p(\mathscr{H}^{\text{triv}}) \otimes_\pi \mathscr{A}) \rtimes G$ and $\mathscr{A}_1 := (\ell^1(\mathscr{H}^{\text{triv}}) \otimes_\pi \mathscr{A}) \rtimes G$. The canonical inclusion $i : \ell^1(\mathscr{H}^{\text{triv}}) \hookrightarrow \ell^p(\mathscr{H}^{\text{triv}})$ extends to $i : \mathscr{A}_1 \longrightarrow \mathscr{A}_p$. We will show it induces a chain-homotopy equivalence by showing that it possesses an inverse at the level of periodic cyclic complexes. By definition of Schatten ideals, it exists an integer $N \geq 0$ such that the image of the multiplication of N elements in $\ell^p(\mathscr{H}^{\text{triv}})$ is bounded in $\ell^1(\mathscr{H}^{\text{triv}})$. It makes commutes the following diagram for all $R > 0$:

$$\begin{tikzcd}
(q_R(\mathscr{A}_p))^N \arrow[r, "j"] \arrow[d, "\text{mult}"'] & q_R(\mathscr{A}_p) \arrow[d, "id\star 0"] \\
\mathscr{A}_1 \arrow[r, "i"'] & \mathscr{A}_p
\end{tikzcd}$$

where j is the inclusion of a higher power of the ideal $q_R(\mathscr{A}_p) \subseteq Q_R(\mathscr{A}_p)$ in itself. It induces a chain-homotopy equivalence due to lemma 8.1.7 applied to $A = \mathscr{A}_p$. The right arrow also induces a chain-homotopy equivalence due to theorem 8.1.8. Write j^{-1} and $(id\star 0)^{-1}$ for their respective chain-homomorphisms inverses. Then it is clear that the composition $mult \circ j^{-1} \circ (id\star 0)^{-1}$ is an inverse of i at the level of periodic cyclic complexes, which ends up the proof. □

Corollary 8.3.3. *The following diagram commutes and the vertical arrows are chain-homotopy equivalences :*

$$\begin{tikzcd}
\widehat{CC}\big((q_R(q_1\mathbb{C}) \otimes_\pi \mathscr{A}) \rtimes G\big) \arrow[r, "(\mathbb{1}_K \otimes id_{\mathscr{A}}) \rtimes G"] \arrow[d, "((id\star 0)^2 \otimes id_{\mathscr{A}}) \rtimes G"', "\sim"] & \widehat{CC}\big((M_2(\ell^{p,G}(\mathscr{H})) \otimes_\pi \mathscr{A}) \rtimes G\big) \\
\widehat{CC}(\mathscr{A} \rtimes G) \arrow[r, "id_{\mathscr{A} \rtimes G}"] & \widehat{CC}(\mathscr{A} \rtimes G) \arrow[u, "\sim", "((1 \mapsto p_1) \otimes id_{\mathscr{A}}) \rtimes G"']
\end{tikzcd}$$

Démonstration. It is clear that the diagram commutes by definition of the morphism $\mathbb{1}_K$ in 5.1.8. The left vertical arrow is a chain-homotopy equivalence by theorem 8.1.8 and the right vertical arrow is also by the lemma above. □

8.3.2 Rotation trick

In this section, we use a form of Atiyah's rotation trick to compute both side composition of the Dirac and dual Dirac element using the flip of coordinates in the manifold $G/K \times G/K$. By simplicity of notation, **during this section we will write $\mathscr{M}$ for the algebra $M_2(\mathscr{C}^\infty_\tau(G/K))$.**

Lemma 8.3.4 (Rotation trick)**.** *Recall the definition of the dual Dirac element β using* *the idempotent matrices e_0 and e_1 2.2.9. The following diagrams commute :*

$$\begin{tikzcd}
q_R^\infty(q_1^\infty \mathscr{M}) \arrow[rr, "q_R^\infty((e_0\otimes id)\star(e_1\otimes id))"] \arrow[d, "q_R^\infty(\Delta_1)"'] & & q_R^\infty(\mathscr{M}\otimes_\pi \mathscr{M}) \arrow[dd, "\nabla_R"] \\
q_R^\infty(q_1\mathbb{C}\otimes_\pi \mathscr{M}) \arrow[d, "\nabla_R"'] & & \\
q_1\mathbb{C}\otimes_\pi q_R^\infty \mathscr{M} \arrow[rr, "\beta^\sharp\otimes id"] & & \mathscr{M}\otimes_\pi q_R^\infty \mathscr{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
q_R^\infty(q_1^\infty \mathscr{M}) \arrow[rr, "q_R^\infty((id\otimes e_0)\star(id\otimes e_1))"] \arrow[d, "q_R^\infty(\nabla_1)"'] & & q_R^\infty(\mathscr{M}\otimes_\pi \mathscr{M}) \arrow[dd, "\nabla_R"] \\
q_R^\infty(\mathscr{M}\otimes_\pi q_1\mathbb{C}) \arrow[d, "\nabla_R"'] & & \\
\mathscr{M}\otimes_\pi q_R(q_1\mathbb{C}) \arrow[rr, "id\otimes q_R^\infty(\beta^\sharp)"] & & \mathscr{M}\otimes_\pi q_R^\infty \mathscr{M}
\end{tikzcd}$$

Démonstration. The proof is straightforward and follows from the definition of Δ_R and ∇_R in 8.1.9. □

Theorem 8.3.5. *Let G be a real Lie group with maximal compact subgroup K such that G/K is even-dimensional. For every complete locally convex G-algebra A, the G-equivariant algebra homomorphism $\alpha^\sharp : q_R^\infty(\mathscr{C}_\tau^\infty(G/K)) \longrightarrow \ell^{p,G}(\mathscr{H})$ coming from the Dirac element $\alpha \in K_G^0(\mathscr{C}_\tau(G/K))$ induces an chain-complex homomorphism which is invertible modulo homotopy*

$$\widehat{CC}\Big((\alpha^\sharp\otimes id_{\mathscr{A}})\rtimes K\Big) : \widehat{CC}\Big((q_R^\infty(\mathscr{C}_\tau^\infty(G/K))\otimes_\pi \mathscr{A})\rtimes K\Big) \stackrel{\sim}{\longrightarrow} \widehat{CC}\Big((\ell^{p,G}(\mathscr{H})\otimes_\pi \mathscr{A})\rtimes K\Big).$$

Démonstration. We will show that this chain-complex homomorphism possesses right and left inverses modulo homotopy. The proposition 8.2.4 states that there exists a K-equivariant homotopy equivalence between the morphisms of algebras :

$$\alpha^\sharp\circ q_R^\infty(\beta^\sharp) : q_R(q_1\mathbb{C}) \longrightarrow M_2(\ell^{p,G}(\mathscr{H})) \text{ and } \mathbb{1}_K : q_R(q_1\mathbb{C}) \longrightarrow M_2(\ell^{p,G}(\mathscr{H})).$$

The corollary 8.3.3 states that the second homomorphism is chain-homotopy equivalent to the identity through the functor $\widehat{CC}((-\otimes_\pi \mathscr{A})\rtimes K)$, which is that $\widehat{CC}\big((\alpha^\sharp\otimes id_{\mathscr{A}})\rtimes K\big)$ is right-invertible of inverse $\widehat{CC}\big((q_R^\infty(\beta^\sharp)\otimes id_{\mathscr{A}})\rtimes K\big)$. It remains to show its left invertibility. Consider the following diagrams :

$$\begin{tikzcd}
q_R^\infty(q_1\mathscr{M}) \arrow[r, "\nabla_R\circ q_R^\infty(\Delta_1)"] \arrow[d, "q_R^\infty((e_0\otimes id)\star(e_1\otimes id))"'] & q_1\mathbb{C}\otimes_\pi q_R^\infty\mathscr{M} \arrow[r, "id_{q_1}\otimes\alpha^\sharp"] \arrow[d, "\beta^\sharp\otimes id_{q_R^\infty}"] & q_1\mathbb{C}\otimes_\pi M_2(\ell^{p,G}(\mathscr{H})) \arrow[d, "\beta^\sharp\otimes id_{M_2}"] \\
q_R^\infty(\mathscr{M}\otimes_\pi\mathscr{M}) \arrow[r, "\nabla_R"'] & \mathscr{M}\otimes_\pi q_R^\infty\mathscr{M} \arrow[r, "id_{\mathscr{M}}\otimes\alpha^\sharp"'] & \mathscr{M}\otimes_\pi M_2(\ell^{p,G}(\mathscr{H}))
\end{tikzcd}$$

$$\begin{tikzcd}
q_R^\infty(q_1\mathscr{M}) \arrow[r, "\nabla_R\circ q_R^\infty(\nabla_1)"] \arrow[d, "q_R^\infty((id\otimes e_0)\star(id\otimes e_1))"'] & \mathscr{M}\otimes_\pi q_R(q_1\mathbb{C}) \arrow[d, "id_{\mathscr{M}}\otimes q_R^\infty(\beta^\sharp)"] & \\
q_R^\infty(\mathscr{M}\otimes_\pi\mathscr{M}) \arrow[r, "\nabla_R"'] & \mathscr{M}\otimes_\pi q_R^\infty\mathscr{M} \arrow[r, "id_{\mathscr{M}}\otimes\alpha^\sharp"'] & \mathscr{M}\otimes_\pi M_2(\ell^{p,G}(\mathscr{H}))
\end{tikzcd}$$

The left squares commute due to the rotation trick 8.3.4 above while the first right square commutes because the coordinates are independent to each others. Moreover, the proposition 2.2.8 states that the idempotent matrices e_0 and e_1 produce morphisms of algebras that are homotopic :

$$(e_i \otimes id) \sim (id \otimes e_i) : \mathscr{M} \longrightarrow \mathscr{M} \otimes_\pi \mathscr{M}, \quad \text{for } i = 0, 1.$$

This homotopy is K-equivariant because the morphism Θ of lemma 2.2.6 applied to $V = \mathscr{C}^\infty_\tau(G/K)$ preserves the isometric action of K. In other words, the vertical left arrows are K-equivariantly homotopic, which makes the following compositions K-equivariantly homotopic to each others by a diagram chasing argument

$$q^\infty_R(q_1\mathscr{M}) \xrightarrow{\nabla_R \circ q^\infty_R(\Delta_1)} q_1\mathbb{C} \otimes_\pi q^\infty_R(\mathscr{M}) \xrightarrow{id_{q_1\mathbb{C}} \otimes \alpha^\sharp} q_1\mathbb{C} \otimes_\pi M_2(\ell^{p,G}(\mathscr{H})) \xrightarrow{\beta^\sharp \otimes id_{M_2}} \mathscr{M} \otimes_\pi M_2(\ell^{p,G}(\mathscr{H})) \tag{39}$$

$$q^\infty_R(q_1\mathscr{M}) \xrightarrow{\nabla_R \circ q^\infty_R(\nabla_1)} \mathscr{M} \otimes_\pi q_R(q_1\mathbb{C}) \xrightarrow{id_{\mathscr{M}} \otimes q^\infty_R(\beta^\sharp)} \mathscr{M} \otimes_\pi q^\infty_R \mathscr{M} \xrightarrow{id_{\mathscr{M}} \otimes \alpha^\sharp} \mathscr{M} \otimes_\pi M_2(\ell^{p,G}(\mathscr{H}))$$

In proposition 8.2.4 we obtained $\alpha^\sharp \circ q^\infty_R(\beta^\sharp)$ as K-equivariantly homotopic to $\mathbb{1}_K$, the last composition then becomes $(id_{\mathscr{M}} \otimes \mathbb{1}_K) \circ (\nabla_R \circ q^\infty_R(\nabla_1))$ modulo K-equivariant homotopy. But the chain-homomorphisms

$$\widehat{CC}\Big((id_{\mathscr{M}} \otimes \mathbb{1}_K \otimes id_{\mathscr{A}}) \rtimes K\Big) \text{ and } \widehat{CC}\Big((\nabla_R \circ q^\infty_R(\nabla_1) \otimes id_{\mathscr{A}}) \rtimes K\Big)$$

are both chain-homotopy equivalent to the identity due to 8.3.3 and 8.1.9. Thus the first composition (39) then is chain-homotopy equivalent to the identity through the functor $\widehat{CC}((- \otimes_\pi \mathscr{A}) \rtimes K)$. We obtained the chain-complex homomorphism $\widehat{CC}((\beta^\sharp \otimes id_{M_2} \otimes id_{\mathscr{A}}) \rtimes K)$ as a left-inverse of $\widehat{CC}((id_{q_1\mathbb{C}} \otimes \alpha^\sharp \otimes id_{\mathscr{A}}) \rtimes K)$, which is chain-homotopy equivalent to $\widehat{CC}((\alpha^\sharp \otimes id_{\mathscr{A}}) \rtimes K)$ by lemma 8.1.8. The morphism $\widehat{CC}((\alpha^\sharp \otimes id_{\mathscr{A}}) \rtimes K)$ being both right-invertible and left-invertible, it ends up the proof. □

8.4 Main theorem

It finally remains to glue all the pieces together. The main argument is the following lemma, which is consequence of the theorem of Nistor [Nis93].

Lemma 8.4.1. *Let A and B be two Banach G-algebras, K a maximal compact subgroup of G, and a G-equivariant algebra homomorphism $\phi : A \to B$. If the induced chain-complex homomorphism*

$$\widehat{CC}(\phi \rtimes K) : \widehat{CC}(A \rtimes K) \longrightarrow \widehat{CC}(B \rtimes K)$$

is invertible modulo homotopy, then $HP_\bullet(\phi \rtimes G) : HP_\bullet(A \rtimes G) \to HP_\bullet(B \rtimes G)$ is an isomorphism.

Démonstration. We can extend the map ϕ into morphisms $\phi \rtimes G : A \rtimes G \to B \rtimes G$ and $\phi \rtimes K : A \rtimes K \to B \rtimes K$ as it is G-equivariant and in particular K-equivariant. The result [Nis93][Theorem 1.1] tells us that for any maximal ideal $\mathfrak{m}$ of the class functions algebra $\mathscr{C}^\infty(G)^{\mathrm{inv}}$ we have the following isomorphism, natural in A :

$$HP_\bullet(A \rtimes G)_{\mathfrak{m}} \simeq HP_{\bullet+\dim(G/K)}(A \rtimes K)_{\mathfrak{m}}.$$

The under-scripts stand as localizations with respect to $\mathfrak{m}$ of these $\mathscr{C}^\infty(G)^{\mathrm{inv}}$-modules. Using this theorem for the G-algebras A and B builds the following commutative diagram with vertical isomorphisms :

$$\begin{tikzcd}
HP_\bullet(A \rtimes G)_{\mathfrak{m}} \arrow[r, "\phi \rtimes G"] \arrow[d, "\sim"'] & HP_\bullet(B \rtimes G)_{\mathfrak{m}} \arrow[d, "\sim"] \\
HP_{\bullet+\dim(G/K)}(A \rtimes K)_{\mathfrak{m}} \arrow[r, "\phi \rtimes K"] & HP_{\bullet+\dim(G/K)}(B \rtimes K)_{\mathfrak{m}}
\end{tikzcd}$$

Indeed, as $\widehat{CC}(\phi \rtimes K)$ is supposed to be invertible modulo homotopy, the bottom arrow becomes an isomorphism, and so is the the top arrow. We showed that $\phi \rtimes G$ is globally defined and a quasi-isomorphism locally, and so is a global quasi-isomorphism. □

Theorem 8.4.2 (Main theorem)**.** *Let G be a reductive group and K be a maximal compact subgroup. For every complete locally convex G-algebra A, the Dirac element $\alpha \in K_G^{\dim(G/K)}(\mathscr{C}_\tau(G/K))$ induces an isomorphism in periodic cyclic homology*

$$(\alpha^\sharp \otimes id_{\mathscr{A}}) \rtimes G : HP_\bullet(\mathscr{A} \rtimes K) \xrightarrow{\sim} HP_{\bullet+\dim(G/K)}(\mathscr{A} \rtimes G)$$

where $\mathscr{A} = A \otimes_\pi \mathscr{D}$ is the stabilized algebra.

Démonstration. Suppose first that G/K is even-dimensional. In that case, the theorem 8.3.5 applies and states that the morphism

$$\alpha^\sharp \otimes id_{\mathscr{A}} : q_R^\infty(\mathscr{C}_\tau^\infty(G/K)) \otimes_\pi \mathscr{A} \longrightarrow M_2(\ell^{p,G}(\mathscr{H})) \otimes_\pi \mathscr{A}$$

verifies the assumptions of the lemma 8.4.1 above. In other words, the Dirac element α realizes an isomorphism of periodic cyclic homology groups :

$$(\alpha^\sharp \otimes id_{\mathscr{A}}) \rtimes G : HP_\bullet\Big((q_R^\infty(\mathscr{C}_\tau^\infty(G/K)) \otimes_\pi \mathscr{A}) \rtimes G\Big) \xrightarrow{\sim} HP_\bullet\Big((M_2(\ell^{p,G}(\mathscr{H})) \otimes_\pi \mathscr{A}) \rtimes G\Big).$$

The theorems 8.1.8 and 7.3.5 states that there exists a chain-homotopy equivalence between the periodic cyclic complexes

$$\widehat{CC}\Big((q_R^\infty(\mathscr{C}_\tau^\infty(G/K)) \otimes_\pi \mathscr{A}) \rtimes G\Big) \xrightarrow{\sim} \widehat{CC}\Big((\mathscr{C}_\tau^\infty(G/K) \otimes_\pi \mathscr{A}) \rtimes G\Big) \xrightarrow{\sim} \widehat{CC}\big(\mathscr{C}_c^\infty(G/K, \mathscr{A}) \rtimes G\big).$$

On the other hand, lemma 8.3.2 shows that morphism $1 \in \mathbb{C} \mapsto p_1 \in M_2(\ell^{p,G}(\mathscr{H}))$ induces the following chain-homotopy equivalence :

$$\widehat{CC}(\mathscr{A} \rtimes G) \xrightarrow{\sim} \widehat{CC}\Big(\big(M_2(\ell^{p,G}(\mathscr{H})) \otimes_\pi \mathscr{A}\big) \rtimes G\Big).$$

In particular, the Dirac element realizes an isomorphism of periodic cyclic homology groups :

$$(\alpha^\sharp \otimes id_{\mathscr{A}}) \rtimes G : HP_\bullet(\mathscr{C}_c^\infty(G/K, \mathscr{A}) \rtimes G) \xrightarrow{\sim} HP_\bullet(\mathscr{A} \rtimes G).$$

Composing this isomorphism with the isomorphism of theorem 7.3.4 applied to the stabilized algebra $\mathscr{A} = A \otimes_\pi \mathscr{D}$ leads to

$$HP_\bullet(\mathscr{A} \rtimes K) \xrightarrow{\sim} HP_\bullet\big((\mathscr{C}_c^\infty(G/K, \mathscr{A}) \rtimes G\big) \xrightarrow{\sim} HP_\bullet(\mathscr{A} \rtimes G).$$

Now, if G/K is odd-dimensional, consider the real reductive group $G' = G \times \mathbb{R}$ whose maximal compact subgroup remains K. In that case, G'/K is even-dimensional and the Thom-isomorphism gives $HP_\bullet(\mathscr{A} \rtimes K) \xrightarrow{\sim} HP_\bullet(\mathscr{A} \rtimes G') \xrightarrow{\sim} HP_{\bullet+1}(\mathscr{A} \rtimes G)$. $\square$

This theorem might be thought as the homological counterpart of the Baum-Connes conjecture with coefficients for Lie groups. It realizes the Dirac induction (31) at the level of periodic cyclic homology :

$$\begin{tikzcd}
K_i(\mathscr{A} \rtimes_\alpha K) \arrow[r, "\text{D-Ind}_{\mathscr{A}}"] \arrow[d, dash, dashed] & K_{i+\dim(G/K)}(\mathscr{A} \rtimes_{\alpha,r} G) \arrow[d, dash, dashed] \\
K_i(\mathscr{A} \rtimes K) \arrow[d, "Ch"'] & K_{i+\dim(G/K)}(\mathscr{A} \rtimes G) \arrow[d, "Ch"] \\
HP_i(\mathscr{A} \rtimes K) \arrow[r, "(\alpha^\sharp \otimes id_{\mathscr{A}}) \rtimes G"'] & HP_{i+\dim(G/K)}(\mathscr{A} \rtimes G)
\end{tikzcd} .$$

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